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General Relativistic Quantum Mechanics: Intrinsic Spin Theory

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Abstract

On a Lorentz-manifold, a mathematically intrinsic spin-theory (meaning that the spin-operators and spin-fields themselves are tensors) is constructed (for arbitrary spin), the reason being that this is what you need in order to compute the generalization of the Einstein equation in general relativity, for particles with spin. Some of the corresponding (total) metric variations are also described, and from those, one can see that it gives rise to stress-energy tensors with positive statistical energy-densities, the latter being necessary in order to create a quantum field theory.

This spin-theory only exists, in its completeness, in a four-dimensional Lorentz metric space, since it depends on the fact that the square of the Hodge star operator is minus one when acting on two-forms.

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1 Introduction[1]

The notion of physically intrinsic spin arises in quantum mechanics in order to explain certain observed phenomena, for instance the splitting of the spectral lines of a hydrogen atom under the influence of a magnetic field (Zeeman effect). The word *intrinsic* here means that it is not explained by properties having spatial dimensions, but only by a property of the associated statistical wave.

Mathematically this is commonly achieved by arbitrarily hooking a spin representation onto the value space for the statistical wave. There results an axiomatic theory, with many details left unexplained. Such constructions are mathematically extrinsic, in the sense that the data are being hooked onto the analysis of the manifold under consideration as additional data, as opposed to a mathematically intrinsic construction, where all data (like operators and statistical waves) are tensors.

If you, for instance, want to compute the generalization of the so called Einstein equation of general relativity, to include particles with physically intrinsic spin, then, as briefly indicated below in this introduction, it is necessary to make use of such a mathematically intrinsic spin theory, in order to enable the needed (total) metric variations. This was the manner the original equation Einstein constructed was extended, by Hilbert, to include electromagnetic interactions: First, the electric field and the magnetic field was unified as an electromagnetic two-form, and secondly, the generalization of the Einstein equation was obtained by a metric variation performed on a suitable modification of the Lagrangian.

Briefly recall that the equations of general relativity can be constructed using a Lagrangian

$$L := s_g + g(F,F) + \sum (g(P_j,P_j) + m_j^2 + e_i g(P_i, A)) \tag{1.1}$$

where s_g is the so called scalar curvature, P_j the energy-momentum of particle j, having rest-mass m_j and e_j being its electric charge; where A is the electromagnetic potential, and where F is the electromagnetic two-form. One will have to put coefficients in front of the terms in this Lagrangian, and these constants will then depend on such things as the physical units used, whether the metric g used is of type $+---$ or $-+++$, the type of variation[2] being performed (metric, in a field, etc.), questions of parametrization[3] (like $P_j = P_j(s)$, where s is the internal time), and how the fields are being assumed to be linked together in the performed variation (for example, one may assume that $dA = F$, or if

[1] Some of the results mentioned in this introduction have not yet been published, particularly those concerning the metric variations; however, this is necessary explanatory material in order for the reader to understand the context.

[2] So this approach may not lead to an immediate unification.

[3] General relativity also uses different Lagrangians, depending on the matter model chosen; this difficulty leads into unsolved general relativistic problems that may only be resolved fully within a combined general relativistic quantum mechanics.

one would speculate on introducing a statistical field ψ, one may allow ψ^{cx} vary as the complex conjugate of ψ or as a separate quantity $\psi_1 = \psi^{\mathrm{cx}}$).

Then, bearing these conditions in mind, the variations are being performed on the integrated Lagrangian

$$\mathcal{L} := L\omega_g \tag{1.2}$$

A (total) metric equation $\delta/\delta g$ results in the so called Einstein equation[4]. A variation $\delta/\delta P_j$ leads to the so called Lorentz world force law

$$\nabla_{P_j} P_j = e_j \tilde{F}(P_j) \tag{1.3}$$

where ∇ is the Levi-Civita connection[5] associated with the metric g, and where $\tilde{F}$ is the two-form F being converted, using the metric g, into a one-one tensor, so it can act on the energy-momentum P_j. And a variation $\delta/\delta A$ with the coupling $F = dA$, leads to the so called Maxwell equations[6]

$$\begin{cases} dA = F \\ \operatorname{div} A = 0 \end{cases} \qquad \begin{cases} dF = 0 \\ \operatorname{div} F = J \end{cases} \qquad \operatorname{div} J = 0 \tag{1.4}$$

where $J = \sum e_j P_j$ is the so called electromagnetic source.

In performing the metric variation $\delta/\delta g$, on the different terms of the Lagrangian $\mathcal{L}$ in equation 1.2, there results, setting $h = \delta g$, expressions of the type

$$\int T(g,h)\omega_g + r(g,h) \tag{1.5}$$

where $r(g,h)/h \to 0$ as $h \to 0$. The expression $T(g,h)$ may still contain some divergencies, but those may be converted into tensors using Stoke's theorem $\int_{\delta R} \alpha = \int_R d\alpha$, for suitable closed regions R and differential forms $\alpha \in \Omega^{\cdot} M$. After this has been done, the resulting tensor $T^g(h)$ may still not be symmetric, but since only symmetric metrics h are under consideration, $T^g(h)$ can be symmetrized[7]; the resulting $T^g(h)$ is being called the stress-energy tensor. If τ

[4] Einstein discovered the resulting equation, without the electromagnetic interactions, by intuitive means. Hilbert simultaneously discovered the same equation using a metric variation, and later extended it to include electromagnetic interactions, by introducing the electromagnetic two-form. See Misner, Thorne & Wheeler [14], for details.

[5] The expression $\nabla_{P_j} P_j$ is called the world acceleration, which is the acceleration of particle j with the gravitation being taken into consideration; a typical general relativistic concept.

[6] The assumption $dF = 0$ is the non-existence of monopoles. The assumption $\operatorname{div} A = 0$ can cause difficulties in theories entirely built up around gauge invariances; however, gauge invariances expresses a form of change-of-observer principle which does not exist in general relativity, and different gauge invariances collide logically with each other, so they are observed symmetries, for which (currently) no unified scheme is known.

[7] This can be motivated by physical reasoning, see Landau & Lifshitz, *Classical Fields* [11], §32.

is an observer, then $T^g(\tau,\tau) := T^g(\tau\otimes\tau)$ is the observed energy density, so one requirement for this to make physical sense is $T^g(\tau,\tau) \geq 0$.

Briefly recall the Dirac equation[8], which says that for a particle with a four-component (complex-valued) statistical field ψ, rest-mass m, electric charge e, and gyro-magnetic[9] ratio $g_{\mathrm{M}} = 2 + \Delta g_{\mathrm{M}}$, one has[10]

$$\mathrm{i}\,\partial\!\!\!/\psi = (m + eA\!\!\!/ - \frac{\Delta g_{\mathrm{M}}}{2}\frac{e}{4m}\,\mathrm{i}\,F\!\!\!/)\psi \tag{1.6}$$

where, as above, A is the electromagnetic potential, and F the electromagnetic field F. Here[11] is the used notation $\partial\!\!\!/ := \gamma^\mu\partial_\mu := \gamma^\mu\frac{\partial}{\partial x^\mu}$, $A\!\!\!/ := \gamma^\mu A_\mu$, $F\!\!\!/ := \frac{1}{2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}$, where γ^μ are some 4×4 complex matrices satisfying the Clifford relations

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \tag{1.7}$$

An easy way to obtain a mathematically intrinsic version of the Dirac equation is as follows: On a four-dimensional Lorentz manifold M, with metric g and quadratic form $q(x) := g(x,x)$, introduce the Clifford bundle[12]

$$\mathrm{C}(g) := \mathrm{T}^{0,\cdot}M/(x\otimes x = q(x)) \tag{1.8}$$

so that $\mathrm{C}(g)$ is the total cotensor algebra bundle on the manifold M divided out, at each stalk, with the relation $x\otimes x = q(x)$. One can also construct $\mathrm{C}(g)$ as being the differential forms algebra bundle $\Omega^\cdot M$ endowed with a different multiplication $c(x) := e(x)+i(x)$, the sum of the exterior and interior multiplications, cf. equation (4.22).

Construct[13,14] the spin representation S as follows: At each stalk $a\in M$, decompose the complexification $V_{\mathbb{C}}$ of the vectorspace $V := \mathrm{T}^*M_a$ of cotensors into the direct sum (moving smoothly) of two (maximal, totally) isotropic vectorspaces $\check{W},\hat{W}$, so that one has $V_{\mathbb{C}} = \check{W}\oplus\hat{W}$. Then, for $x\in\Omega^1$, define an action onto the exterior algebra $S := \Omega^\cdot_{\mathbb{C}}\check{W}$ by

$$\gamma(x) := \sqrt{2}(e(\check{x}) + i(\hat{x})) \tag{1.9}$$

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[8] In this equation, the speed of light c and the Heisenberg constant $\hbar$ are both set to one, and the metric is assumed to be of type $+---$.

[9] Electrons and muons have gyro-magnetic ratio g_{M} very close to 2, but protons have $g_{\mathrm{M}} = 5.59$ and neutrons have $g_{\mathrm{M}} = -3.83$. See Itzykson & Zuber [8], p16.

[10] This equation appears in Itzykson & Zuber [8], equation (2-74); they use the notation $\sigma^{\mu\nu}F_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} = \mathrm{i}\,F\!\!\!/$.

[11] Using the commutator $[a,b] := ab - bc$ and anticommutator $\{a,b\} := ab + ba$ notations.

[12] For details see §4.5.

[13] For details about this construction, or rather the variation using $\gamma(x) := e(x) + 2i(x)$, see Bourbaki, *Algebra* [4], chIX, §9.

[14] In using a positive definite metric, instead of a Lorentz metric, a similar construction just results in an almost complex structure. This works in any even dimension of the manifold M, and it leads simply to complex analysis of that manifold.

where $\check{x}$ and $\hat{x}$ denotes the projections onto the vectorbundles $\check{W}$ and $\hat{W}$, respectively. Since $\gamma(x)^2 = q(x)$, this extends to a Clifford bundle representation, which is in fact the usual Dirac spin representation[15].

Plugging in elements, this spin representation may in fact be identified with the so called chiral (χ-ral) representation that physicists use, in the following manner: With respect to a restricted orthonormal[16] basis ω^μ, set

$$(1.10) \qquad \begin{cases} \check{a} := \frac{1}{\sqrt{2}}(dt + dz) \\ \check{b} := \frac{1}{\sqrt{2}}(dx - \mathrm{i}\, dy) \end{cases} \qquad \begin{cases} \hat{a} := \frac{1}{\sqrt{2}}(dt - dz) \\ \hat{b} := \frac{1}{\sqrt{2}}(dx + \mathrm{i}\, dy) \end{cases}$$

Then set $\check{W} := (\check{a}, \check{b})_{\mathbb{C}}$, $\hat{W} := (\hat{a}, \hat{b})_{\mathbb{C}}$, and give the spin representation space $S := \Omega^{\cdot}_{\mathbb{C}}\check{W}$ the basis $(1, \check{a} \wedge \check{b}; \check{a}, \check{b})$. Working this out in coordinates[17], using the given basises, one then finds,

$$(1.11) \qquad \gamma^\mu := \gamma(dx^\mu) := \begin{pmatrix} 0 & \sigma_\mu \\ \sigma^\mu & 0 \end{pmatrix}$$

where the matrices σ^μ are the usual Pauli spin matrices

$$(1.12) \qquad \sigma^0 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \sigma^3 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad \sigma^1 := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad \sigma^2 := \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$$

In other words, there results the chiral representation[18].

In order to obtain positive probabilities, physicists use an[19] invariant $j^0\psi := \psi^\dagger\psi$, where $\psi^\dagger$ denotes complex-transpose in the chosen matrix basis, associated with any Dirac spinor $\psi \in S = \Omega^{\cdot}_{\mathbb{C}}\check{W}$. In the basises chosen in order to obtain equation (1.11), one can see[20] that

$$(1.13) \qquad \psi^\dagger\psi' = \hat{g}_\tau(\psi^{\mathrm{cx}}, \psi')$$

[15]Keeping in mind that $A, F \in \Omega^{\cdot} = \mathrm{C}(g)$, one can then write $\not{A} = \gamma(A)$ and $\not{F} = \gamma(F)$; this equality depends on the tensor formula (4.12) below, since it says that $\gamma(F) = \frac{1}{2}\gamma(F_{\mu\nu}dx^\mu \wedge dx^\nu) = \frac{1}{2}\gamma(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu)F_{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} = \not{F}$.

[16]Here, like in the rest of this paper, we set $dx^0 :=: cdt$, $dx^1 :=: dx$,$dx^2 :=: dy$,$dx^3 :=: dz$, with c being the speed of light, often set equal to 1.

[17]In these formulas, equations (1.11) and (1.13), the metric g is assumed to be of type $+---$.

[18]By a chiral representation is here meant any one with matrices of the form $\varepsilon^\mu\gamma^\mu$, where γ^μ are the matrices in equation (1.11), and where ε^μ is a row of signs $\pm$, which depends on which convention you are using. By a suitable modification of the choice of basis elements for the vector bundles $\check{W}, \hat{W}$ and the spin representation S, any choice of such a row of signs ε^μ can be made to appear.

[19]In fact there is a whole host of invariants, described both in physics literature, like Itzykson & Zuber [8], and mathematical papers, like Brauer & Weyl, Amer.J.Math. 57(1935), p425-429. However, the question under consideration here, is to understand how a positive definite metric can be made appear from constructions in general relativity.

[20]All one has to do in order to verify this, is to compute $\hat{g}_\tau(\check{a}^{\mathrm{cx}}, \check{a}) = 1$, and $\hat{g}_\tau(\check{b}^{\mathrm{cx}}, \check{b}) = 1$, which follows, keeping in mind that $\hat{g}_\tau(dx^\mu, dx^\nu) = \delta^{\mu\nu}$; it then also follows that $\hat{g}_\tau((\check{a} \wedge \check{b})^{\mathrm{cx}}, \check{a} \wedge \check{b}) = 1$.

for any two Dirac spinor fields $\psi, \psi' \in S = \Omega^{\cdot}_{\mathbb{C}}\check{W} \subset \Omega^{\cdot}_{\mathbb{C}}M$, where ψ^{cx} now denotes (the mathematically intrinsic) complex conjugation of ψ considered as being a tensor on the manifold M, and where $\hat{g}_\tau$ denotes the positive definite metric obtained by a suitable reflection in the time coordinate $\tau := dt$. A formula for $\hat{g}_\tau$ is

$$\hat{g}_\tau(x, y) = 2g(x, \tau)g(\tau, y) - g(x, y)q(\tau) \tag{1.14}$$

if $x, y \in \Omega^1 M$, and for general elements $\psi, \psi' \in \Omega^{\cdot}_{\mathbb{C}}M$, one uses linear extension.

The interesting thing is now that it is possible to extricate this positive definite metric $\hat{g}_\tau(\alpha, \beta)$, where $\alpha, \beta \in \Omega^{\cdot}_{\mathbb{C}}M$, as an observed statistical energy-density, arising from a stress-energy tensor of general relativity.

We will make use of a (total) metric variation on the Lagrangian

$$L = g^{-1}(\alpha, \beta) \tag{1.15}$$

where g^{-1} indicates that it is really is the matrix inverse of the metric that we are making use of: Since $\delta g^{-1} = -g^{-1}\,\delta g\, g^{-1}$, it is important in situations involving a metric variation to keep tensors and cotensors apart; shifting tensors also causes a sign change. Let ω_g denote the metric volume element (cf. equation (5.28)).

Then the formula[21] is, for $\alpha, \beta \in \Omega^p_{\mathbb{C}}M$,

$$-\frac{\delta g^{-1}(\alpha, \beta)\omega_g}{\delta g}.(\tau, \tau) = \tfrac{1}{2}(-q(\tau))^{p-1}\hat{g}_\tau(\alpha, \beta)\omega_g \tag{1.16}$$

Here one may write

$$\frac{\delta g^{-1}(\alpha, \beta)\omega_g}{\delta g} = T^{\alpha,\beta}\omega_g \tag{1.17}$$

where (up to a sign), $T^{\alpha,\beta}(x, y)$ is the associated stress-energy tensor.

There are two conclusions to form from this:

First, as was already mentioned above, the needed positive probabilities of quantum mechanics, arise as observed statistical energy-densities of general relativity, computed via a general relativistic stress-energy tensor from a (total) metric variation. From the practical, mathematical, point of view, this means that as long as one is being concerned with observed quantum mechanical data, there is no problem in obtaining the positive definite metric needed for the evaluation of the theory — you simply invoke the concepts dictionary of general relativity.

The second conclusion to form, is that if we are hoping for developing a genuinely general relativistic statistical wave equation for particles with physically

[21] In the special case $\alpha = \beta = F$, the electromagnetic two-form, this metric variational formula reduces to a formula for the electromagnetic stress-energy tensor, showing that the electromagnetic energy-densities are non-negative.

intrinsic spin, then we need to first develop a suitable mathematically intrinsic spin theory, because otherwise we cannot perform the needed metric variations — and it is here the contents of this paper comes into place.

The Clifford bundle construction above for the Dirac spinors is not suitable for this purpose, mainly because it links bi-spin types $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ strongly together; above we had to rely on a direct sum decomposition $V_{\mathbb{C}} = \check{W} \oplus \hat{W}$ into two (maximal, totally) isotropic vectorspace bundles $\check{W}$ and $\hat{W}$. We would need to find a suitable $\mathrm{o}(g)$ action onto single (maximal, totally) isotropic vectorspace bundles. One obvious attempt might be to go down to the even subalgebra $\mathrm{C}(g)^{\mathrm{even}}$ of $\mathrm{C}(g)$, which acts on the vectorbundle $\check{W} \subset S = \Omega^{\cdot}_{\mathbb{C}}\check{W}$; however, this does not work, since the defined action in equation (1.9) still involvs the complementary isotropic bundle $\hat{W}$. Further the formulas of this $\mathrm{C}(g)^{\mathrm{even}}$ action onto $\check{W}$, when written out explicitly, in my experience, yield nothing, and is wholly unworkable.

So one has to proceed differently:

In section 2 a new kind of spin operators, written out in coordinates, are being introduced. Then in section 3 the (maximal totally) isotropic vectorspaces onto which these spin operators act, are being classified; this corresponds to a classification under the orbits of the real, restricted Lorentz group, as opposed to the one using the complex, proper Lorentz group, which you achieve using pure spinors. The number of orbits, two, is the same, but the new thing here is the geometric description with respect to the (real) Lorentz data, necessary for a suitable physics description.

Sections 4 and 5 just contains some review of the tensor formulas involved, needed for attaining a suitable coordinate independent description of the spin operators introduced in a coordinate setting in section 2, and for which I have found no other convenient reference.

Section 6 then contains the coordinate independent description of the spin operators constructed in section 2. There is also shown that these spin operators can be constructed as the even Clifford algebra bundle $\mathrm{C}(g)^{\mathrm{even}}$ acting on the split-up odd subbundles ${}_{\pm}\mathrm{C}(g)^{\mathrm{odd}}$, where the splitting is being caused by the idempotents $\frac{1}{2}(1 \pm \mathrm{i}\,\omega) \in \mathrm{C}(g)$, with ω, as usual, being the metric volume element.

Section 7 (not included in this preprint version) then describes the isomorphism corresponding to the classical (axiomatic) tensor product isomorphism $(\frac{1}{2},0) \otimes (0,\frac{1}{2}) \cong (\frac{1}{2},\frac{1}{2})$ between bispin types $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$. Among other things, in the Lorentz context under consideration, this isomorphism turns out to depend on the presence of a pre-chosen time coordinate, of unclear physical interpretation to me at the time of this writing. Another feature is, even though one in principle can dismiss of Clifford algebra techniques (despite being convenient) in describing the general spin theory, in describing this isomorphism, Clifford algebra techniques turn out to be the right thing.

2 The Spin Operators; Coordinate Version

In order to conveniently describe spin actions in Lorentz-metric, I will here below introduce spin operators $\sigma^j_\pm$. With a suitable choice of isotropical coordinates, these will be seen to correspond exactly to the Pauli spin-matrices $[\sigma^j]^{\mathrm{cx}}$, and σ^j, respectively. The construction is also such, that it is independent of whether the metric is of type $+---$, or type $-+++$; it turns out that this is the most convenient way to handle the two different metric types simultaneously.

So let M be a Lorentz-manifold, with metric g, of type $+---$ or $-+++$, and if p is a point of M, then V stands for the cotangent space M^*_a at this point p. Also write $q(x) := g(x,x)$ for the quadric form q associated with g. If M, at this point p, has a local coordinate basis e_μ, with local coordinate coordinate functions x^μ, then V has the corresponding basis $\omega^\mu := dx^\mu$. An element in V then is of the form $a = a_\mu dx^\mu$, or for short a_μ. If the local basis e_μ is subject to a local transformation, arbitrarily termed covariant, then x^μ and dx^μ transform with inverses, or contravariantly, and the coordinates a_μ of $a \in V$ transform with inverses to that, so they become covariant.

For the rest of this section, we can from the mathematical point of view forget about the origin of V as the cotangent vectorspace at a point p of M, and simply think of it as a vectorspace endowed with a metric g, but I have found it intuitive to keep its origin in mind. In effect, V is a Minkowski space.

The basis ω^μ above will always be a *restricted* orthonormal basis, restricted meaning it is oriented, and ω^0 is futurepointing; then ω^j is space-oriented.

Let (j,k,l) be an even permutation of the numbers 1,2 and 3, that is $(1,2,3)$, $(2,3,1)$, or $(3,1,2)$. Now, with respect to a restricted orthonormal tetrad ω^μ, the spin operators $\sigma^j_\pm : V_\mathbb{C} \to V_\mathbb{C}$ are defined by

$$(2.1) \qquad \sigma^j_\pm := \begin{cases} \omega^0 \to \omega^j \\ \omega^j \to \omega^0 \end{cases} \quad \begin{cases} \omega^k \to \mp \mathrm{i}\, \omega^l \\ \omega^l \to \pm \mathrm{i}\, \omega^k \end{cases}$$

when operating on basis elements, and for other elements of V, by linear extension.

The reasons for chosing this given $\pm$ sign convention, are that $\exp(\mathrm{i}\,\theta\sigma^j_\pm)$ is then a $\binom{\text{positive}}{\text{negative}}$ rotation ($\theta > 0$) on the vectorspace (ω^k, ω^l) generated by ω^k and ω^l, and that the vectors-spaces $\mathrm{W}^\pm_\gamma := (\omega^0 + \omega^3, \omega^1 \pm \mathrm{i}\,\omega^2)_\mathbb{C}$, $\gamma := \omega^0 + \omega^3$, defined and studied more in detail below[22], are being invariant under the operators $\sigma^j_\pm$. Unfortunately, the most commonly used Pauli spin matrices

$$(2.2) \qquad \sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then becomes associated with the negative spin operators σ^j_-, but I have found it better to not reverse the signs in these other two situations indicated.

[22]Cf. equation (2.6) and section 3.

It is easy to see, that the spin operators $\sigma^j_\pm$ are elements of the Lie-algebra $\mathrm{o}(g)$ of the metric g, meaning that for any $x, y \in V_\mathbb{C}$, one has $g(\sigma^j_\pm.x, y) + g(x, \sigma^j_\pm.y) = 0$. This can easily be seen by direct verification; indeed, by linearity, it is enough to verify that $g(\sigma^j_\pm.\omega^\nu, \omega^\mu) = -g(\sigma^j_\pm.\omega^\mu, \omega^\nu)$, and one sees directly from the definition of the operators $\sigma^j_\pm$ that it is enough to verify this in the two cases $\mu, \nu \in \{0, j\}$ and $\mu, \nu \in \{k, l\}$, because if not, both sides of the equation are zero; and in these two cases it is easy. Alternatively, one can observe, as is done below[23], that the operators $\sigma^j_\pm$ can be constructed as (with ε_σ being the sign of the spatial part of the metric g) the two-forms $\varepsilon_\sigma(\omega^0 \wedge \omega^j \pm \mathrm{i}\,\omega^k \wedge \omega^l)$ being converted to Lie-algebra deformations using the metric g, and so this automatically yield elements in $\mathrm{o}(g)$.

To mathematically specify what is defined over the real numbers and what is defined over the complex numbers turns out to be a tricky business. Strictly speaking, the operators $\sigma^j_\pm \in \mathrm{o}(g)_\mathbb{C}$, but these operators are really the projections from the real Lie algebra $\mathrm{o}(g)$ via some complex-valued projection operators, namely the operators $\frac{1}{2}(1 \mp \mathrm{i}\,*)$, where $*$ is the Hodge star operator acting on two-forms. From the physical point of view $\mathrm{o}(g)$ corresponds to angular momentum or electromagnetic infinitesimal deformations, in other words, observables, which are always real. Mathematically, this does not give a situation of pure rationality, with real objects acting on real objects, and complex objects acting on complex objects, but a mixed rationality, with some real objects acting on complex objects.

Some straight-forward computations show that the spin operators $\sigma^j_\pm$ satisy the following relations:

2.1 Spinoperator relations.
Below, (j, k, l) is an even permutation of $(1, 2, 3)$.

$$\text{(2.3)} \qquad \begin{cases} (\sigma^j_\pm)^2 = 1 \\ \sigma^j_\pm \sigma^k_\pm = \pm\,\dfrac{1}{\mathrm{i}}\,\sigma^l_\pm \end{cases} \qquad \begin{cases} \{\sigma^j_\pm,\, \sigma^k_\pm\} = 0, \textit{for } j \neq k \\ [\sigma^j_+,\, \sigma^{j'}_-] = 0, \textit{for any } j,\ j' \end{cases}$$

Here $\{a, b\} = ab + ba$ is the anti-commutator, and $[a, b] = ab - ba$ denotes the commutator.

Note that if one wants to have the relation $\sigma^j_\pm \sigma^k_\pm = \pm\,\mathrm{i}\,\sigma^l_\pm$ instead of the one indicated, and still having the $\sigma^j_\pm$ leaving $(dt + dz, dx \pm \mathrm{i}\,dy)_\mathbb{C}$ invariant, one method to attain that is by writing the functions to the right, like $(x)\sigma^j_\pm$, instead of to the left.

The fact that σ^j_+ and $\sigma^{j'}_-$ commute for any j, j', corresponds, as we shall see[24], to the decomposition of $\mathrm{o}(g) \cong \mathrm{sl}_2(\mathbb{C})$ into spin/antispin types $(\frac{1}{2}, 0)$ and

[23] See equation (5.23).
[24] See formulas (6.14) and (6.26).

$(0, \frac{1}{2})$. Mathematically, in the model presented here below[25], the result also derives from the fact that the idempotents $\frac{1}{2}(1 \mp \mathrm{i}\,*)$ commute, where $*$ is the Hodge star operator acting on differential two-forms.

It will be necessary to compute the action of the operators $\sigma^j_\pm$ on the various (maximal, totally) isotropic vectorspaces of $V_\mathbb{C}$ explicitly, in terms of matrices acting on certain basises, and for that purpose, we introduce some notation.

In the kind of explicit computations that is to follow, it is more explicit to label the basis-elements $\omega^\mu = dx^\mu$, $\mu = 0, 1, 2, 3$, as dt, dx, dy, and dz (remembering the origin of V as the cotangent stalk at a point of a Lorentz-manifold). Since in physics $dx^0 = c\,dt$, where c is the speed of light, this notation involvs setting $c = 1$.

Then write, with respect to any restricted orthonormal basis ω^μ,

$$\begin{cases} \gamma^\pm := dt \pm dz \\ w^\pm := dx \pm \mathrm{i}\,dy \end{cases} \tag{2.4}$$

and it will be convenient to write $\gamma := \gamma^+ = dt + dz$ for short. The elements $\gamma^\pm$ and $w^\pm$ are isotropic; an element x is called isotropic if $q(x) = 0$, and if x is real this is what some call "null". One also has the relations $g(\gamma^\varepsilon, w^\eta) = 0$, for $\varepsilon, \eta \in \{+1, -1\}$, and

$$\begin{cases} \frac{1}{2} g(\gamma^+, \gamma^-) = \varepsilon_\tau \\ \frac{1}{2} g(w^+, w^-) = \varepsilon_\sigma \end{cases} \qquad \begin{cases} \varepsilon_\tau := \text{sign of temporal part of } g \\ \varepsilon_\sigma := \text{sign of spatial part of } g \end{cases} \tag{2.5}$$

Then, again with respect to any restricted orthonormal basis ω^μ, and with $\gamma := dt + dz$ as above, define

$$\mathrm{W}^\pm_\gamma := (dt + dz, dx \pm \mathrm{i}\,dy)_\mathbb{C}. \tag{2.6}$$

The $\mathrm{W}^\pm_\gamma$ are then maximal isotropic complex vectorspaces; a vectorspace will here be called *isotropic*[26] if, for any x, y in it, $g(x, y) = 0$. Since the metric g is non-degenerate, one can see that an isotropic complex vectorspace contained in $V_\mathbb{C}$ is maximal exactly when it has complex dimension two.

It will be shown below[27] that, as the ω^μ run through all restricted orthonormal basises, the above defined vectorspaces $\mathrm{W}^\pm_\gamma$ exhaust all maximal isotropic vectorspaces: This corresponds to a classification of the maximal isotropic vectorspaces under the action of the real restricted Lorentz group $\mathrm{L}^\uparrow_+$, as opposed to pure spinor theory of Clifford algebras, which only gives a classification under

[25] See section 6.

[26] There is also a terminology in use, where a vectorspace is called isotropic if merely there is a non-zero element with zero quadratic form value in it; what is here called isotropic is then called totally isotropic. Instead of the word isotropic, some use one of the words "singular" or "null"; some reserve the word "null" to be used for real elements, those that are on the light-cone.

[27] See proposition 3.2.

the proper complex Lorentz group $\mathrm{L}^+_{\mathbb{C}}$. Further, each maximal isotropic vectorspace W intersects the light-cone in a unique line on the light-cone[28], which may then be represented by a non-zero element γ (future-pointing if you so wish), and through this light-cone line, there are exactly two maximal isotropic vectorspaces[29], namely, W^+_γ and W^-_γ, thereby explaining our notation.

One can think of each of the vectorspaces $\mathrm{W}^\pm_\gamma$ as a spinor realization; from the point of physics this corresponds to a reduction of the degeneracy of order two (dimension two eigenspaces) of the operators $\sigma^j_\pm$ acting on $V_{\mathbb{C}}$ (which constitute a representation of the spin states), to pure states (one dimensional eigenspaces).

We will return to these topics further down in the text, in §6.1, where they, together with some additional facts will be demonstrated.

Now, if (dt, dx, dy, dz) is a restricted orthonormal basis, so is $(dt, dx, -dy, -dz)$, and therefore $\mathrm{W}^\pm_{\gamma^-} = (dt - dz, dx \pm \mathrm{i}(-dy))_{\mathbb{C}} = (dt - dz, dx \mp \mathrm{i}\, dy)_{\mathbb{C}}$. Keeping the notation from above, this gives

$$\text{(2.7)} \qquad \begin{cases} \mathrm{W}^+_{\gamma^\pm} = (\gamma^\pm, w^\pm)_{\mathbb{C}} \\ \mathrm{W}^-_{\gamma^\pm} = (\gamma^\pm, w^\mp)_{\mathbb{C}} \end{cases} \qquad \begin{cases} \mathrm{W}^\pm_{\gamma^+} = (\gamma^+, w^\pm)_{\mathbb{C}} \\ \mathrm{W}^\pm_{\gamma^-} = (\gamma^-, w^\mp)_{\mathbb{C}} \end{cases}$$

which also can be written

$$\text{(2.8)} \qquad \mathrm{W}^\eta_{\gamma^\varepsilon} = (\gamma^\varepsilon, w^{\varepsilon\eta})_{\mathbb{C}}, \qquad \text{for } \varepsilon, \eta \in \{+1, -1\}.$$

In the following tables, which are done by means of explicit computations only, the vectorspaces $\mathrm{W}^\eta_{\gamma^\varepsilon}$, for $\varepsilon, \eta \in \{+1, -1\}$, are also given the basises $(\gamma^\varepsilon, w^{\varepsilon\eta})$:

$$\text{(2.9)} \quad \sigma^3_\pm : \begin{cases} dt \to dz \\ dz \to dt \end{cases} \begin{cases} dx \to \mp\mathrm{i}\, dy \\ dy \to \pm\mathrm{i}\, dx \end{cases} \quad \sigma^3_+ : \begin{cases} \gamma^\pm \to \pm\gamma^\pm \\ w^\pm \to \mp w^\pm \end{cases} \quad \sigma^3_- : \begin{cases} \gamma^\pm \to \pm\gamma^\pm \\ w^\pm \to \pm w^\pm \end{cases}$$

$$\text{(2.10)} \qquad \sigma^3_+ = \pm\sigma^3 \quad \text{on } \mathrm{W}^+_{\gamma^\pm} \qquad \sigma^3_- = \pm\sigma^3 \quad \text{on } \mathrm{W}^-_{\gamma^\pm}$$

$$\text{(2.11)} \quad \sigma^1_\pm : \begin{cases} dt \to dx \\ dx \to dt \end{cases} \begin{cases} dy \to \mp\mathrm{i}\, dz \\ dz \to \pm\mathrm{i}\, dy \end{cases} \quad \sigma^1_+ : \begin{cases} \gamma^\pm \to w^\pm \\ w^\pm \to \gamma^\pm \end{cases} \quad \sigma^1_- : \begin{cases} \gamma^\pm \to w^\mp \\ w^\pm \to \gamma^\mp \end{cases}$$

$$\text{(2.12)} \qquad \sigma^1_+ = \sigma^1 \quad \text{on } \mathrm{W}^+_{\gamma^\pm} \qquad \sigma^1_- = \sigma^1 \quad \text{on } \mathrm{W}^-_{\gamma^\pm}$$

$$\text{(2.13)} \quad \sigma^2_\pm : \begin{cases} dt \to dy \\ dy \to dt \end{cases} \begin{cases} dz \to \mp\mathrm{i}\, dx \\ dx \to \pm\mathrm{i}\, dz \end{cases} \quad \sigma^2_+ : \begin{cases} \gamma^\pm \to \mp\mathrm{i}\, w^\pm \\ w^\pm \to \pm\mathrm{i}\, \gamma^\pm \end{cases} \quad \sigma^2_- : \begin{cases} \gamma^\pm \to \pm\mathrm{i}\, w^\mp \\ w^\pm \to \pm\mathrm{i}\, \gamma^\mp \end{cases}$$

$$\text{(2.14)} \qquad \sigma^2_+ = \mp\sigma^2 \quad \text{on } \mathrm{W}^+_{\gamma^\pm} \qquad \sigma^2_- = \pm\sigma^2 \quad \text{on } \mathrm{W}^-_{\gamma^\pm}$$

[28]See proposition 3.1.
[29]See the statement 3.6.

Inspection of the table above shows that the operators σ_+^j leave the vectorspace W_γ^+ invariant, and similarly, the operators σ_-^j leave the vectorspace W_γ^- invariant. We summarize:

2.2 Summary. *Given a restricted orthonormal basis* ω^μ*, the operators* $\sigma_\pm^j$*,* $j = 1, 2, 3$*, leave the vectorspace* $\mathrm{W}_\gamma^\pm = (dt + dz, dx + \mathrm{i}\, dy)_\mathbb{C}$ *invariant, and with the same elements as basis for* $\mathrm{W}_\gamma^\pm$*, one has*

$$\begin{cases} \sigma_+^j = (\sigma^j)^{\mathrm{cx}} & \textit{on } \mathrm{W}_\gamma^+ = (\gamma, w^+)_\mathbb{C} = (dt + dz, dx + \mathrm{i}\, dy)_\mathbb{C} \\ \sigma_-^j = \sigma^j & \textit{on } \mathrm{W}_\gamma^- = (\gamma, w^-)_\mathbb{C} = (dt + dz, dx - \mathrm{i}\, dy)_\mathbb{C} \end{cases} \tag{2.15}$$

Note that there is a catch here, because the operators $\sigma_\pm^j$ and the vectorspaces $\mathrm{W}_\gamma^\pm$ are defined in terms of the *same* restricted orthonormal basis ω^μ. However, this difficulty is removed, as we below[30] shall see, by passing to coordinate-free descriptions: The linear span of the operators $\sigma_\pm^j$ over $\mathbb{R}$ may be identified with the vectorspace of holomorphic/antiholomorhic elements under the action of the Hodge $*$ operator acting on the differential two-forms, and as remarked above (and as will be demonstrated below, cf. the statement 3.6), there are exactly two maximal isotropic vectorspaces through the line on the light-cone defined by γ, namely W_γ^+ and W_γ^-. Since the linear span of the operators $\sigma_\pm^j$, in this identification, is independent of the basis chosen, one sees that it leaves $\mathrm{W}_\gamma^\pm$ fixed by representing $\mathrm{W}_\gamma^\pm = (dt + dz, dx \pm \mathrm{i}\, dy)_\mathbb{C}$ in a suitable restricted orthonormal basis ω^μ; it then follows from the above table (or summary 2.2).

Also note that the operators $\sigma_\mp^j$ do not leave the vectorspaces $\mathrm{W}_\gamma^\pm$ invariant; in fact, $\exp(\mathrm{i}\,\theta\sigma_\mp^j)$, where one can take $\theta \in \mathbb{R}$ if you so please, acts transitively on the vectorspaces $\mathrm{W}_\gamma^\pm$, which follows from the next following table.

First note that $\exp(\theta\sigma_\pm^j)$ is easy to compute, since $(\sigma_\pm^j)^2 = 1$. Because of this, one has

$$\begin{aligned} \exp(\theta\sigma_\pm^j) &= \sum_{n=0}^{\infty} \frac{1}{n!}\,(\theta\sigma_\pm^j)^n = \Big(\sum_{n \text{ even}} \frac{1}{n!}\,\theta^n\Big)\cdot 1 \;+\; \Big(\sum_{n \text{ odd}} \frac{1}{n!}\,\theta^n\Big)\,\sigma_\pm^j \\ &= (\cosh\theta)\cdot 1 \;+\; (\sinh\theta)\,\sigma_\pm^j \end{aligned} \tag{2.16}$$

Written out, the actions $\exp(\theta\sigma_\pm^j)\,\omega^\mu$ then become, for an even permutation (j, k, l) of $(1, 2, 3)$,

$$\begin{aligned} &\begin{cases} \exp(\theta\sigma_\pm^j)\,\omega^0 = \cosh\theta\,\omega^0 + \sinh\theta\,\omega^j \\ \exp(\theta\sigma_\pm^j)\,\omega^j = \sinh\theta\,\omega^0 + \cosh\theta\,\omega^j \end{cases} \\ &\begin{cases} \exp(\theta\sigma_\pm^j)\,\omega^k = \cosh\theta\,\omega^k \mp \mathrm{i}\sinh\theta\,\omega^l \\ \exp(\theta\sigma_\pm^j)\,\omega^l = \pm\,\mathrm{i}\sinh\theta\,\omega^k + \cosh\theta\,\omega^l \end{cases} \end{aligned} \tag{2.17}$$

[30] See section 6.

In order to get the corresponding equations for $\exp(\mathrm{i}\,\theta\sigma^j_\pm)$, use the identities

$$\left\{\begin{aligned}\cosh(\mathrm{i}\,\theta) &= \cos\theta\\ \sinh(\mathrm{i}\,\theta) &= \mathrm{i}\sin\theta\end{aligned}\right. \qquad \left\{\begin{aligned}\cos(\mathrm{i}\,\theta) &= \cosh\theta\\ \sin(\mathrm{i}\,\theta) &= \mathrm{i}\sinh\theta\end{aligned}\right. \tag{2.18}$$

The action of $\exp(\theta\sigma^j_\pm)$ on the isotropical elements $\gamma^\pm$ and $w^\pm$ then becomes

$$\left\{\begin{aligned}\exp(\theta\sigma^3_+)\,\gamma^\pm &= \mathrm{e}^{\pm\theta}\,\gamma^\pm\\ \exp(\theta\sigma^3_+)\,w^\pm &= \mathrm{e}^{\mp\theta}\,w^\pm\end{aligned}\right. \tag{2.19}$$

$$\left\{\begin{aligned}\exp(\theta\sigma^3_-)\,\gamma^\pm &= \mathrm{e}^{\pm\theta}\,\gamma^\pm\\ \exp(\theta\sigma^3_-)\,w^\pm &= \mathrm{e}^{\pm\theta}\,w^\pm\end{aligned}\right. \tag{2.20}$$

$$\left\{\begin{aligned}\exp(\theta\sigma^1_+)\,\gamma^\pm &= \cosh\theta\,\gamma^\pm + \sinh\theta\,w^\pm\\ \exp(\theta\sigma^1_+)\,w^\pm &= \sinh\theta\,\gamma^\pm + \cosh\theta\,w^\pm\end{aligned}\right. \tag{2.21}$$

$$\left\{\begin{aligned}\exp(\theta\sigma^1_-)\,\gamma^\pm &= \cosh\theta\,\gamma^\pm + \sinh\theta\,w^\mp\\ \exp(\theta\sigma^1_-)\,w^\pm &= \sinh\theta\,\gamma^\mp + \cosh\theta\,w^\pm\end{aligned}\right. \tag{2.22}$$

$$\left\{\begin{aligned}\exp(\theta\sigma^2_+)\,\gamma^\pm &= \cosh\theta\,\gamma^\pm \mp \mathrm{i}\sinh\theta\,w^\pm\\ \exp(\theta\sigma^2_+)\,w^\pm &= \pm\mathrm{i}\sinh\theta\,\gamma^\pm + \cosh\theta\,w^\pm\end{aligned}\right. \tag{2.23}$$

$$\left\{\begin{aligned}\exp(\theta\sigma^2_-)\,\gamma^\pm &= \cosh\theta\,\gamma^\pm \pm \mathrm{i}\sinh\theta\,w^\mp\\ \exp(\theta\sigma^2_-)\,w^\pm &= \pm\mathrm{i}\sinh\theta\,\gamma^\mp + \cosh\theta\,w^\pm\end{aligned}\right. \tag{2.24}$$

It is already clear from the table above that, say $\exp(\theta\sigma^j_-)$ does not leave the vectorspace $\mathrm{W}^+_{\gamma^\pm} = (\gamma^\pm, w^\pm)_{\mathbb{C}}$ invariant, except if $j = 3$, which can be interpreted as that the antispin axis of the vectorspace $\mathrm{W}^+_{\gamma^\pm}$ is along the z-axis; one should then note that we here deal with a representation of spin type $(\frac{1}{2}, \frac{1}{2})$, and that later[31] the action of one of the sets of operators $\sigma^j_\pm$ will be set to zero in order to get the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ spin type representations.

The action of the operators $\sigma^j_\mp$ on the vectorspace $\mathrm{W}^\pm_\gamma$ will now be described more in detail.

First note that

$$\sigma^j_\pm(\gamma\wedge w^\pm) = 0, \quad \text{Lie action,} \tag{2.25}$$

which corresponds for the Pauli spin matrices σ^j to the fact that $\operatorname{tr}\sigma^j = 0$. Here *Lie action* of an operator A on a product $a\wedge b$ is $A(a\wedge b) := (Aa)\wedge b + a\wedge(Ab)$.

[31] See §6.1.

Also, for an even permutation (j,k,l) of $(1,2,3)$, define the operator J^j by

$$J^j : \begin{cases} \omega^0 \to 0 \\ \omega^j \to 0 \end{cases} \quad \begin{cases} \omega^k \to \omega^l \\ \omega^l \to -\omega^k \end{cases} \tag{2.26}$$

so J^j is the generator for infinitesimal positive rotations about the j-axis.

Then

$$\exp(\mathrm{i}\,\theta\sigma^j_{\mp})(\gamma \wedge w^{\pm}) = \exp(\mp 2\theta J^j)(\gamma \wedge w^{\pm}), \quad \text{group action.} \tag{2.27}$$

Here, the *group action* of an operator A on a product $a \wedge b$ is $A(a \wedge b) := (Aa) \wedge (Ab)$.

Proof: First note that $\mathrm{i}(\sigma^j_{\mp} - \sigma^j_{\pm}) = \mp 2J^j$, and since $\sigma^j_{\pm}(\gamma \wedge w^{\pm}) = 0$ and $[\sigma^j_+, \sigma^j_-] = 0$; one gets

$$\begin{aligned}\exp(\mathrm{i}\,\theta\sigma^j_{\mp})(\gamma \wedge w^{\pm}) &= \exp(\mathrm{i}\,\theta(\sigma^j_{\mp} - \sigma^j_{\pm}))(\gamma \wedge w^{\pm}) \\ &= \exp(\mp 2\theta J^j)(\gamma \wedge w^{\pm})\end{aligned}$$

It is now possible to explicitly describe how the operators $\exp(\mathrm{i}\,\theta\sigma^j_{\mp})$ move the vectorspaces $\mathrm{W}^{\pm}_{\gamma}$:

First recall that there is a 1-1 bijection between non-zero decomposable two-forms $\alpha \in \Omega^2 V_{\mathbb{C}}$ equivalent up to a complex scalar, and two dimensional complex subspaces W of $V_{\mathbb{C}}$. A two-form α is here called *decomposable* if it can be written on the form $\alpha = a \wedge b$, for some $a, b \in V_{\mathbb{C}}$. If (a,b) is a basis of W, then $\alpha := a \wedge b$ defines a decomposable two-form, unique up to complex scalar, since if (a', b') is another basis of W, then there is an invertible complex linear transformation A such that $(a', b') = A(a,b)$, and then $a' \wedge b' = \det A\, a \wedge b$.

On the other hand, if a non-zero two-form α is decomposable, it can be written $\alpha = a \wedge b$, and the vectorspace $W := (a,b)_{\mathbb{C}}$ can be retrieved from α by the formula $W = \{x \in V_{\mathbb{C}} \mid x \wedge \alpha = 0\}$, independently of the basis chosen for W. (Incidentally, this argument also shows that if $\alpha = a \wedge b = a' \wedge b'$ are two decompositions of α, then there is a invertible complex linear transformation A such that $(a', b') = A(a,b)$, since the indicated formula for W shows that the two pairs of elements a, b and a', b' generate the same vector-space, namely W.)

Returning to how the operators $\exp(\mathrm{i}\,\theta\sigma^j_{\mp})$ move the vectorspaces $\mathrm{W}^{\pm}_{\gamma}$, the vectorspace $\mathrm{W}^{\pm}_{\gamma}$ becomes, in the correspondence described above, associated with the two-form $\gamma \wedge w^{\pm}$. But we also have

$$\begin{aligned}\exp(\mathrm{i}\,\theta\sigma^j_{\mp})(\gamma \wedge w^{\pm}) &= \exp(\mp 2\theta J^j)(\gamma \wedge w^{\pm}) \\ &= (\exp(\mp 2\theta J^j)\gamma) \wedge (\exp(\mp 2\theta J^j) w^{\pm})\end{aligned} \tag{2.28}$$

Now, $\exp(\mp 2\theta J^j)$ is a restricted Lorentz-transformation, sending the restricted orthonormal basis ω^{μ} to another restricted orthonormal basis, but also leaving

dt fixed (it is a space rotation about the j-axis), and $\exp(\mp 2\theta J^j)\gamma$ is an element on the light-cone, so it follows that

$$\exp(\mathrm{i}\,\theta\sigma^j_\mp).\mathrm{W}^\pm_\gamma = \mathrm{W}^\pm_{\gamma'}, \quad \textit{where } \gamma' = \exp(\mp 2\theta J^j).\gamma \tag{2.29}$$

(And the one who doubts this, should become convinced by writing out the actions explicitly.) It should be emphasized that this equality of vectorspaces is an equality of sets, not elements.

It can be worthwile to give a criterion for a two-form α to be decomposable, making use of the Lorentz metric g. For this purpose, with the notation $\tilde\omega^j := \omega^k\wedge\omega^l$ for an even permutation (j,k,l) of $(1,2,3)$, write $\alpha := a_j\,\omega^0\wedge\omega^j + b_j\,\tilde\omega^j = \omega^0\wedge\mathbf{a} + \mathbf{b}\cdot\tilde{\boldsymbol{\omega}}$. Then we have the following:

2.3 *The two form α is decomposable exactly when*

$$g(\alpha, *\alpha) = 0, \quad \textit{or equivalently, when} \quad \mathbf{a}\cdot\mathbf{b} = 0 \tag{2.30}$$

where $*$ is the Hodge star operator.

Proof: In fact, this is just the Grassman[32] condition $\alpha_{01}\alpha_{23} + \alpha_{02}\alpha_{31} + \alpha_{03}\alpha_{12} = 0$ for a two-form $\alpha = \sum_{\mu<\nu}\alpha_{\mu\nu}\,\omega^\mu\wedge\omega^\nu$. One has $*\alpha = -a_j\tilde\omega^j + b_j\omega^0\wedge\omega^j$, so $g(\alpha,*\alpha) = a_jb_jq(\omega^0\wedge\omega^j) - a_jb_jq(\tilde\omega^j) = -2a_jb_j = -2\mathbf{a}\cdot\mathbf{b}$, and setting this to zero is the same thing as the Grassman condition.

However, one can prove this directly, too, which is good, because it gives a method to decompose two-forms whenever it is possible to do so. Suppose first that α is decomposable, say $\alpha = u\wedge v$. Then $g(\alpha,*\alpha)\omega = \alpha\wedge **\alpha = -\alpha\wedge\alpha$, and this is zero, since it is equal to $(u\wedge v)\wedge(u\wedge v)$, so $g(\alpha,*\alpha) = 0$.

On the other hand, assume that $\mathbf{a}\cdot\mathbf{b} = 0$. If $\mathbf{a} = 0$, then write arbitrarily $\mathbf{b} = \mathbf{u}\times\mathbf{v}$, with $\mathbf{u},\mathbf{v}$ being some spatial vectors (that is, orthogonal to ω^0), and then $\mathbf{u}\wedge\mathbf{v} = (\mathbf{u}\times\mathbf{v})\cdot\tilde{\boldsymbol{\omega}} = \mathbf{b}\cdot\tilde{\boldsymbol{\omega}} = \alpha$. So assume $\mathbf{a}\neq\mathbf{0}$; then, writing $\tau := \omega^0$, and using the identity $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$,

$$\begin{aligned}(\tau + \frac{1}{\mathbf{a}^2}\mathbf{a}\times\mathbf{b})\wedge\mathbf{a} &= \tau\wedge\mathbf{a} + (\frac{1}{\mathbf{a}^2}(\mathbf{a}\times\mathbf{b})\times\mathbf{a})\cdot\tilde{\boldsymbol{\omega}} \\ &= \tau\wedge\mathbf{a} + \frac{1}{\mathbf{a}^2}((\mathbf{a}\cdot\mathbf{a})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{a})\cdot\tilde{\boldsymbol{\omega}} \\ &= \tau\wedge\mathbf{a} + \mathbf{b}\cdot\tilde{\boldsymbol{\omega}} = \alpha,\end{aligned} \tag{2.31}$$

giving a decomposition of α. From this, we get the following result:

2.4 Proposition. *A two-form α can be written on the form $\alpha = u\wedge v$, where u is time-like (future-pointing if you so wish) and v is space-like if and only if the following two conditions hold:*

i. $g(\alpha,*\alpha) = 0$

ii. $q(\alpha) < 0$

[32] See for instance Bourbaki *Algebra* [4], ch3, §11.13, formula (85), p611.

Proof: If α is of the indicated form $\alpha = u \wedge v$, then α is decomposable, and therefore $g(\alpha, *\alpha) = 0$. If, in addition, u is time-like and v is space-like, then one can replace v by its component orthogonal to u, namely $v_u^\perp := v - (g(u,v)/q(u))\, u$, since $u \wedge v = u \wedge v_u^\perp$. Since $q(v_u^\perp) = q(v) - (g(u,v)^2/q(u))$, one sees, checking separately the cases where the metric g is of types $+---$ and $-+++$, that also $v_u^\perp$ is spatial. Thus $q(u \wedge v) = q(u \wedge v_u^\perp) = q(u)\, q(v_u^\perp) < 0$.

Conversely, if $g(\alpha, *\alpha) = 0$, then α is decomposable, and can be written on the form $\alpha = u \wedge v$.

Suppose first that $q(u) = q(v) = 0$. Then

$$0 > q(\alpha) = \det \begin{pmatrix} q(u) & g(u,v) \\ g(v,u) & q(v) \end{pmatrix} = -g(u,v)^2,$$

so $g(u,v) \neq 0$. Since then $q(u \pm v) = \pm 2g(u,v)$, one of the two elements $u + v$ and $u - v$ is timelike, and the other one is spacelike, and since further $\alpha = u \wedge v = \frac{1}{2}(u - v) \wedge (u + v)$, this gives the indicated type of decomposition of α.

Suppose on the other hand that not both of $q(u)$ and $q(v)$ are zero, say $q(u) \neq 0$. Then, as above, replace v by $v_u^\perp$, the part orthogonal to u. This gives $0 > q(\alpha) = q(u \wedge v) = q(u \wedge v_u^\perp) = q(u)\, q(v_u^\perp)$, and so one of the elements u and $v_u^\perp$ is time-like, and the other one is space-like.

If further, we have $\alpha = u \wedge v$, with u time-like and v space-like, and u happens to be past-pointing, then reverse the signs of u and v to obtain a decomposition where u is future-pointing.

The interest of proposition 2.4 above is as follows: If you want to write the vectorspace $V_{\mathbb{C}}$ as a direct sum of two maximal (complex) isotropic vectorspaces W and W', as $V_{\mathbb{C}} = W \oplus W'$, the only possibility (as we shall see below) is $V_{\mathbb{C}} = \mathrm{W}_\gamma^\pm \oplus \mathrm{W}_{\gamma'}^\pm$ for a choice of sign $\pm$, and where γ and γ' are elements defining distinct lines on the light-cone (equivalently $g(\gamma, \gamma') \neq 0$). Then $\alpha := \gamma \wedge \gamma'$ defines a real two-form, unique up to a real non-zero scalar, and α satisfies the conditions of the proposition: The first condition is clearly satisfied, since α is decomposable, and the two elements $\gamma \pm \gamma'$ are orthogonal and one of them is time-like and the other is space-like, so $q(\alpha) = q(\frac{1}{2}(\gamma - \gamma') \wedge (\gamma + \gamma')) = \frac{1}{4} q(\gamma - \gamma') q(\gamma + \gamma') < 0$, which is the second condition fulfilled.

Now if we decompose α as $\alpha = u \wedge v$ then $(u,v)_{\mathbb{R}}$, the real vectorspace generated u and v, is independent of the decomposition made (because it can be defined as $\{x \in V \,|\, x \wedge \alpha = 0\}$); it intersects the light-cone in exactly two lines, namely those defined by γ and γ' (because we can take u and v equal to γ and γ'). On the other hand, if we take u to be time-like and v space-like, and further assume them to be normalized, then $u + v$ and $u - v$ defines two elements on the light-cone. From proposition 2.4 above one therefore sees that pairs of distinct lines on the light-cone are classified by real two-forms, given up to a scalar, and satisfying the conditions of the proposition.

We summarize:

2.5 Summary. *The set of direct sum decompositions $V_{\mathbb{C}} = W \oplus W'$ of the vectorspace $V_{\mathbb{C}}$ into maximal isotropic vectorspaces W and W' (and with the order of these vectorspaces disregarded) is in 1-1 correspondence with the set of pairs $(\pm, \alpha)$, where $\pm$ is a choice of sign, and α is a real two-form, $\alpha \in \Omega^2 V$, defined up to a real non-zero scalar, and satisfying $q(\alpha, *\alpha) = 0$ and $q(\alpha) < 0$. The correspondence is given by the following connection:*

$$\begin{cases} V_{\mathbb{C}} = \mathrm{W}^{\pm}_{\gamma} \oplus \mathrm{W}^{\pm}_{\gamma'} \\ \alpha = \gamma \wedge \gamma' \end{cases} \tag{2.32}$$

where γ, γ' are two elements defining distinct lines on the light-cone (equivalently γ, γ' are both on the light-cone and $g(\gamma, \gamma') \neq 0$); these two lines on the light-cone are the intersection of the real plane $\{x \in V \mid x \wedge \alpha = 0\}$ and the light-cone.

Then, returning to the interest of such a result, a direct sum decomposition $V_{\mathbb{C}} = W \oplus W'$ of the vectorspace $V_{\mathbb{C}}$ into maximal isotropic vectorspaces W and W' is exactly what you need in order to define a Dirac (four) spinor representation over the complex Clifford algebra $C(g)_{\mathbb{C}}$ (or Clifford bundle, if you let it move on a Lorentz manifold); the description 2.5 above then also takes into account the real data linked to the Lorentz-metric, which is of course indispensible for the complete picture (be it from the mathematical or physical point of view).

3 Isotropic Vectorspaces

The maximal isotropic (complex) vectorspaces contained in $V_{\mathbb{C}}$ are classified below.

Each such isotropic vectorspace is of complex dimension two, and each such space constitute a reduction of the degeneracy of one of the sets, given by a choice of sign $\pm$, of the operators $\sigma^j_\pm$ to pure states; in mathematical terms, the eigenspaces are reduced from complex dimension two to one, becoming an irreducible representation of spin type $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$. Each such isotropic vectorspace is thus a model for the spinors of either type $(\frac{1}{2}, 0)$ or of type $(0, \frac{1}{2})$. Such a representation is sometimes called a semi-spinor or a Weyl spinor, to be distinguished from the Dirac spinors arising from the Dirac equation (which are irreducible representations over the complex Clifford algebra $C(g)_{\mathbb{C}}$, and of complex dimension four).

From pure spinor theory[33] we already know something about the classification of the maximal isotropic vectorspaces. (The reader who does not know anything about pure spinors, and does not want to know about it, can skip over this part, because it is not needed, since we will make another classification below.) Without going into the technical details of this machinery, the outcome is that the set of maximal isotropic subspaces fall in two classes, which here are denoted by $\mathcal{W}^{\pm}$; these two classes are in fact the orbits under the proper complex Lorentz group $\mathrm{L}^{+}_{\mathbb{C}}$. If two distinct isotropic vectorspaces belong to the same class, both in $\mathcal{W}^{+}$ or both in $\mathcal{W}^{-}$, their intersection is zero. On the other hand, if they belong to different classes, their intersection is always of complex dimension one.

We need more details and sophistication than this pure spinor theory gives, for instance, we need to know the orbits of the maximal isotropical vectorspaces under the restriced real Lorentz group $\mathrm{L}^{\uparrow}_{+}$, we need explicit descriptions of the maximal isotropic vectorspaces related to the Lorentz metric, and the construction of the spin-representation over the Clifford-algebra involvs an unwanted assumption of a decomposition $V_{\mathbb{C}} = W \oplus W'$ of $V_{\mathbb{C}}$ into a a direct sum of two maximal isotropical subspaces W and W'. Further, the spin theory of the Clifford-bundle does not at all lead to the spin theory we are going to construct below. So for reasons like this, we have to proceed differently.

Before stating proposition 3.1 below, first note that it will not be needed to distinguish between vectorspaces of real dimension one and real lines if they already are on the real light-cone $\mathcal{L}$, since such real lines clearly pass through zero.

3.1 Proposition. *If W is a maximal isotropic (complex) vectorspace (contained in $V_{\mathbb{C}}$) then its intersection with the (real) light-cone $\mathcal{L}$ is non-zero, $W \cap \mathcal{L} \neq 0$; in fact this intersection is then a real line on $\mathcal{L}$.*

[33] See for instance Chevalley [5], or Benn & Tucker [2].

Proof: If $W \cap \mathcal{L} \neq 0$, then $W \cap \mathcal{L}$ is a real non-zero isotropic vectorspace, of real dimension at most one, since the Lorentz metric g is of index one, that is, a real line. So show that $W \cap \mathcal{L} \neq 0$ then. In fact, I will give two proofs, because they involve different ideas.

First proof: This first proof is interesting, because it gives a method to, via some somewhat mysterious trigonometrical identities, explicitly find the element on the light-cone.

Since W is a maximal isotropic vectorspace, it is of the form $W = (u, v)_{\mathbb{C}}$, where $u, v \in V_{\mathbb{C}}$ are basis elements satisfying $q(u) = q(v) = g(u, v) = 0$. Now, in a given restricted orthonormal ω^μ, these elements have the coordinates u_μ and v_μ, and one of the coordinates u_0 or v_0 must be non-zero, because otherwise W would be a dimension two complex isotropical vectorspace contained in the subspace of $V_{\mathbb{C}}$ orthogonal to ω^0, and where the restriction of the metric g is non-degenerate, which is impossible. (The dimension of an isotropical vectorspace with respect to a non-degenerate metric is at most one half the dimension of the vectorspace it is contained in.)

So we can assume that say $u_0 \neq 0$, and multiplying u with the scalar $(u_0)^{-1}$, we can assume that $u_0 = 1$. Then, replacing v by $v - (v_0)u$, we can assume that $v_0 = 0$. Now, one of the coordinates v_j must be non-zero, and so by a possible rotation of the spatial coordinates, we can assume that $v_1 \neq 0$, and further, by multiplying v by $(v_1)^{-1}$, we can assume that $v_1 = 1$. Then, replacing u by $u - (u_1)v$, we can assume that $u_1 = 0$.

After these straightforward reductions, the elements u and v have the coordinates $u = (1, 0, u_2, u_3)$ and $v = (0, 1, v_2, v_3)$. In addition, the relations $q(u) = q(v) = g(u, v) = 0$, which still hold because they are true for any pair of elements in W, give

$$\begin{cases} (u_2)^2 + (u_3)^2 = 1 \\ (v_2)^2 + (v_3)^2 = -1 \end{cases}$$

and in addition $u_2 v_2 + u_3 v_3 = 0$. In other words, we must have $(u_2, u_3) = (\cos\alpha, \sin\alpha)$ and $(v_2, v_3) = \mathrm{i}(\cos\beta, \sin\beta)$, where $\alpha, \beta \in \mathbb{C}$ are complex angles with $0 = u_2 v_2 + u_3 v_3 = \mathrm{i}(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \mathrm{i}\cos(\alpha - \beta)$, or $\beta = \alpha + \frac{\pi}{2} + n\pi$, where $n \in \mathbb{Z}$, that is, $(v_2, v_3) = \mathrm{i}\,\varepsilon(-\sin\alpha, \cos\alpha)$, for a sign $\varepsilon = \pm 1$.

The task is now to show that for a suitable complex constant $\lambda \in \mathbb{C}$, the element

$$u + \lambda v = (1, \lambda, u_2 + \lambda v_2, u_3 + \lambda v_3) =: (1, \lambda, A, B) \tag{3.1}$$

is real, in other words, $\lambda \in \mathbb{R}$, and $\operatorname{Im} A = \operatorname{Im} B = 0$, giving the equations

$$\begin{cases} 0 = \operatorname{Im} A = \operatorname{Im}\cos\alpha - \lambda\varepsilon \operatorname{Re}\sin\alpha \\ 0 = \operatorname{Im} B = \operatorname{Im}\sin\alpha + \lambda\varepsilon \operatorname{Re}\cos\alpha \end{cases} \tag{3.2}$$

Here we have to verify that these two linear equations do have a solution, that is, if $\operatorname{Re}\sin\alpha = 0$ also $\operatorname{Im}\cos\alpha = 0$, and if $\operatorname{Re}\cos\alpha = 0$ then also $\operatorname{Im}\sin\alpha = 0$, and in addition, we have to show that the equations have a common solution, that is, if both $\operatorname{Re}\sin\alpha \neq 0$ and $\operatorname{Re}\cos\alpha \neq 0$, the two λ produced by both equations are the same.

Now, if we set $\alpha = a + \mathrm{i}\, b$, $a, b \in \mathbb{R}$, the equations

$$\begin{cases} \cos\alpha = \cos(a+\mathrm{i}\, b) = \cos a \cosh b - \mathrm{i} \sin a \sinh b \\ \sin\alpha = \sin(a+\mathrm{i}\, b) = \sin a \cosh b + \mathrm{i} \cos a \sinh b \end{cases} \tag{3.3}$$

show that if $0 = \operatorname{Re}\cos\alpha = \cos a \cosh b$, then $\cos a = 0$, and so $\operatorname{Im}\sin\alpha = \cos a \sinh b = 0$, and similarly, if $0 = \operatorname{Re}\sin\alpha = \sin a \cosh b$, then $\sin a = 0$, and so $\operatorname{Im}\sin\alpha = -\sin a \sinh b = 0$. So the two equatons always produce solutions.

Now one has

$$\operatorname{Im}\cos\alpha \operatorname{Re}\cos\alpha = -\cos a \sin a \cosh b \sinh b = -\operatorname{Re}\sin\alpha \operatorname{Im}\sin\alpha, \tag{3.4}$$

which is what you need, in order to show that the two equations are compatible, since if $\operatorname{Re}\sin\alpha \neq 0$ and $\operatorname{Re}\cos\alpha \neq 0$, what you have to show is

$$\frac{\operatorname{Im}\cos\alpha}{\operatorname{Re}\sin\alpha} = \varepsilon\lambda = -\frac{\operatorname{Im}\sin\alpha}{\operatorname{Re}\cos\alpha}, \tag{3.5}$$

but this then follows.

Second proof: This second proof, which uses spin matrices, is simpler, but the method to successively transform the basis elements of W is the same. This proof also gives a method to explicitly find a non-zero element on the light-cone, but using spin matrix computations instead of trigonometry.

We use the correspondence, utilizing the Pauli spin matrices σ^μ,

$$\begin{cases} k = k_\mu \omega^\mu \rightarrow \tilde{k} := k_\mu \sigma^\mu \\ \quad V \longrightarrow \mathrm{M}_2(\mathbb{C})^{\mathrm{H}} \\ \quad V_{\mathbb{C}} \longrightarrow \mathrm{M}_2(\mathbb{C}) \end{cases} \tag{3.6}$$

which are isomorphisms from V to the set of complex Hermitian 2×2-matrices, and from $V_{\mathbb{C}}$ to the set of complex 2×2-matrices. For the quadratic form q, one has $q(k) = \det \tilde{k}$, and so we can assume that $W \subset \mathrm{M}_2(\mathbb{C})$, with $\det W = 0$, still of complex dimension two, and we want to show that W contains a non-zero Hermitian matrix, which corresponds to a non-zero element on the real light-cone $\mathcal{L}$.

So assume that $W = (u, v)_{\mathbb{C}}$ with basis elements

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad v = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

and $ad = bc$, $a'd' = b'c'$. Then one can assume that at least one of the elements a, d, a', d' is non-zero, because otherwise $bc = 0$ and $b'c' = 0$, and the only way to make W two dimensional is that it contains $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, but then $\det(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)) \neq 0$, so W would not be isotropic. Therefore, one of the elements a, d, a', d' is non-zero, and we can assume it is a or d, otherwise interchange u and v. Further, by a possible reflection $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$, applied to all elements of W, which preserves isotropy and the Hermitian property, we can assume $a \neq 0$, and by replacing u by $a^{-1}u$, we can in addition assume $a = 1$.

Then, by replacing v by $v - a'u$, we can assume that $a' = 0$, and after that, $b'c' = a'd' = 0$ gives that $b' = 0$ or $c' = 0$. By a possible transformation $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \to \left(\begin{smallmatrix} a' & c' \\ b' & d' \end{smallmatrix}\right)$, applied to all elements of W, which preserves the Hermitian property and the other assumptions on u and v, we can assume that $c' = 0$ (alternatively, work the case $b' = 0$ separately).

Here $b' \neq 0$, because if $b' = 0$, first, by replacing v by $(d')^{-1}v$ (if $d' = 0$ too, then W would be one dimensional only), we can assume $d' = 1$, and then by replacing u by $u - dv$, we can assume $d = 0$; from this it then follows $bc = ad = 0$. But then $\det(u + v) = 1 - bc = 1$, contradicting the fact that W was assumed to be isotropic.

Since $b' \neq 0$, by replacing u by $u - b(b')^{-1}v$, we can assume that $b = 0$. Then $ad = bc = 0$, and since $a = 1$, we have $d = 0$, so $u = \left(\begin{smallmatrix} 1 & 0 \\ c & 0 \end{smallmatrix}\right)$.

After these transformations, we thus have two basis elements u, v of W, which are of the form

$$u = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 & b' \\ 0 & d' \end{pmatrix}$$

We arrive here at two cases, namely $d' = 0$, and $d' \neq 0$.

Case I: If $d' = 0$, then, by replacing v with $(b')^{-1}v$, assume $v = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and since then $\det(u + v) = -c$, one gets $c = 0$. In other words $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, which is a non-zero Hermitian matrix in W.

Case II: If $d' \neq 0$, replace v by $(d')^{-1}v$, so that $v = \left(\begin{smallmatrix} 0 & b' \\ 0 & 1 \end{smallmatrix}\right)$. Then $0 = \det(u + v) = \det\left(\begin{smallmatrix} 1 & b' \\ c & 1 \end{smallmatrix}\right)$, so that $cb' = 1$, that is $b' = c^{-1}$, and $v = \left(\begin{smallmatrix} 0 & c^{-1} \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$u + |c|^2 v = \begin{pmatrix} 1 & \bar{c} \\ c & |c|^2 \end{pmatrix} \tag{3.7}$$

is a non-zero Hermitian matrix in W.

Once that we know that every maximal isotropic vectorspace W intersects the (real) light-cone in a line, the rest of the classification is not very difficult to attain.

3.2 Proposition. *i. If W is a maximal isotropic vectorspace, then there is a restricted orthonormal basis ω^μ in which*[34]

$$W = (dt + dz, dx + \mathrm{i}\,\varepsilon dy)_{\mathbb{C}} \tag{3.8}$$

[34]The physical constant the speed of light c is set $c = 1$ in this, and subsequent formulas; if you want it back, simply replace $dt \to c\,dt$.

where $\varepsilon = \pm 1$.

ii. If W*, a maximal isotropic vectorspace, in a given restricted orthonormal basis* ω^μ *is of the form* $W = (dt + dz, v)_{\mathbb{C}}$*, then it can be written*

$$W = (dt + dz, dx + \mathrm{i}\,\varepsilon dy)_{\mathbb{C}} \tag{3.9}$$

where $\varepsilon = \pm 1$*, in this given basis.*

iii. If W*, in a given restricted ortonormal basis* ω^μ*, is of the form* $W = (dt + dz, dx + \mathrm{i}\,\varepsilon dy)_{\mathbb{C}}$*, then the sign* ε *can be computed intrinsically as follows: Take non-zero* $\gamma \in W \cap \mathcal{L}$*, and* $\alpha \in W$ *satisfying* $g(\alpha^{\mathrm{cx}}, \alpha) = \varepsilon_\sigma$*; such an* α *always exist, take* $\alpha = \frac{1}{\sqrt{2}}(dx + \mathrm{i}\,\varepsilon dy)$*, for instance.*

Then

$$\gamma \wedge \alpha^{\mathrm{cx}} \wedge \alpha = \sqrt{-1}\,\varepsilon\varepsilon_\tau\, i(\gamma)\omega \tag{3.10}$$

where $i(\gamma)\omega$ *denotes interior multiplication of* γ *onto* ω*. In addition, both sides of the equation (3.10) are non-zero.*

In particular, since the both sides of the equation (3.10) are non-zero, the sign ε *depends only on the vectorspace* W*, and not on the restricted orthonormal basis chosen.*

Proof: We start proving statement i, by building such an indicated restricted orthonormal basis ω^μ. Take a non-zero $\gamma \in W \cap \mathcal{L}$. If ω^0 is any normalized, timelike, future-pointing vector, first rescale γ so that $g(\omega^0, \gamma) = \varepsilon_\tau$. Then $\omega^3 := \gamma - \omega^0$ is a spatial, normalized element. If we then arbitrarily complete to a restricted, orthonormal basis, the result follows, if we only can prove statement *ii* of the proposition. So proceed with *ii* then.

In order to prove statement *ii*, then, first note that the indicated basis element v of W can, by replacing it with $v - v_0(dt + dz)$, be assumed to have the coordinate $v_0 = 0$, and then also $v_3 = 0$, since $0 = g(v, dt + dz) = \varepsilon_\sigma v_3$. The relation $q(v) = 0$ then gives $(v_1)^2 + (v_2)^2 = 0$, so that $v_2 = \pm\,\mathrm{i}\,v_1$, and so by replacing v with $(v_1)^{-1}v$, we see that v can be taken to be $dx \pm \mathrm{i}\,dy$.

Now to the proof of statement iii. Write $u := dt + dz$, $v := \frac{1}{\sqrt{2}}(dx + \mathrm{i}\,\varepsilon dy)$, so that $u^{\mathrm{cx}} = u$, and $g(v^{\mathrm{cx}}, v) = \varepsilon_\sigma$. Then from the conditions on γ and α it follows that $\gamma = \lambda u$, $\lambda \in \mathbb{R}^\times$, and $\alpha = \mu u + \nu v$, $\mu, \nu \in \mathbb{C}$, with $|\nu| = 1$, since $g(\alpha^{\mathrm{cx}}, \alpha) = |\nu|^2 \varepsilon_\sigma$.

Explicit computations now gives on the one hand $\gamma \wedge \alpha^{\mathrm{cx}} \wedge \alpha = \lambda u \wedge |\nu|^2 v^{\mathrm{cx}} \wedge v = \lambda u \wedge \mathrm{i}\,\varepsilon dx \wedge dy$.

On the other hand, one has $\omega = \frac{1}{2}(dt - dz) \wedge (dt + dz) \wedge dx \wedge dy = \frac{1}{2}(dt - dz) \wedge u \wedge dx \wedge dy$, and $i(u)\frac{1}{2}(dt - dz) = \frac{1}{2}g(dt + dz, dt - dz) = \varepsilon_\tau$, and also $i(u)$ is giving zero if applied to the elements $dt + dz$, dx and dy, so it follows that $i(u)\omega = \varepsilon_\tau u \wedge dx \wedge dy$, and so $i(\gamma)\omega = \lambda\varepsilon_\tau u \wedge dx \wedge dy$.

Then $\sqrt{-1}\,\varepsilon\varepsilon_\tau\, i(\gamma)\omega = \mathrm{i}\,\varepsilon\lambda u \wedge dx \wedge dy = \gamma \wedge \alpha^{\mathrm{cx}} \wedge \alpha$, which is what we were supposed to show.

Clearly also $u \wedge dx \wedge dy \neq 0$, and since $\lambda \neq 0$, both sides of the equation (3.10) are non-zero.

Introduce the following notation:

3.3 *If γ is a non-zero element on the light-cone $\mathcal{L}$, and in a restricted orthonormal basis ω^μ one has $\gamma := dt + dz$, write*

$$\mathrm{W}^{\pm}_{\gamma} := (dt + dz, dx \pm idy)_{\mathbb{C}}. \tag{3.11}$$

From statement ii of proposition 3.2 above, it follows that $\mathrm{W}^{\pm}_{\gamma}$, for a fixed sign $\pm$, only depends on the line on the light-cone $\mathcal{L}$ defined by γ, because in a given restricted orthonormal basis ω^μ, any maximal isotropic vectorspace W is of the form $W = (dt + dz, dx \pm \mathrm{i}\,dy)_{\mathbb{C}}$, so this justifies the notation. One can therefore also speak of $\mathrm{W}^{\pm}_{\gamma}$, where γ is a (non-zero) equivalence class $\gamma \in \mathcal{L}/\mathbb{R}^\times$ (which may also be identified with the set of future-pointing rays), if one so prefers, and this gives a 1-1 correspondence between the maximal isotropic vectorspaces in $V_{\mathbb{C}}$ and the set $\{+1, -1\} \times (\mathcal{L}/\mathbb{R}^\times - 0)$.

It follows from the proposition 3.2 above that the set of maximal isotropic subspaces of $V_{\mathbb{C}}$ form exactly two distinct orbits under the action of the real restricted Lorentz group $\mathrm{L}^{\uparrow}_{+}$: First, if W and $\hat{W}$ are maximal isotropic vectorspaces, for which there is a restricted Lorentz-transformation $\varLambda \in \mathrm{L}^{\uparrow}_{+}$ with $\hat{W} = \varLambda(W)$, then, if $W = (\omega^0 + \omega^3, \omega^1 + \mathrm{i}\,\varepsilon\omega^2)_{\mathbb{C}}$ in some restricted orthonormal basis ω^μ, then clearly $\hat{W} = (\hat{\omega}^0 + \hat{\omega}^3, \hat{\omega}^1 + \mathrm{i}\,\varepsilon\hat{\omega}^2)_{\mathbb{C}}$, in the restricted orthonormal basis $\hat{\omega}^\mu := \varLambda(\omega^\mu)$, so it follows that W and $\hat{W}$ have the same sign ε.

In other words:

3.4 *For restricted Lorentz-transformations $\varLambda \in \mathrm{L}^{\uparrow}_{+}$*

$$\varLambda(\mathrm{W}^{\pm}_{\gamma}) = \mathrm{W}^{\pm}_{\varLambda.\gamma} \tag{3.12}$$

Secondly, if we now would want to prove that for two maximal isotropic vectorspaces W and $\hat{W}$ with the same sign, there is a restricted Lorentz-transformation $\varLambda \in \mathrm{L}^{\uparrow}_{+}$ with $\hat{W} = \varLambda(W)$, it suffices to remark, by formula (3.12) above, that for any pair of lines on the light-cone $\mathcal{L}$, there is such a $\varLambda$ sending the one line to the other. Alternatively, there is a restriced orthonormal basis ω^μ in which $W = (\omega^0 + \omega^3, \omega^1 + \mathrm{i}\,\varepsilon\omega^2)_{\mathbb{C}}$, and there is another restricted orthonormal basis $\hat{\omega}^\mu$ in which $\hat{W} = (\hat{\omega}^0 + \hat{\omega}^3, \hat{\omega}^1 + \mathrm{i}\,\varepsilon\hat{\omega}^2)_{\mathbb{C}}$, so the restricted Lorentz-transformation $\varLambda$ sending ω^μ to $\hat{\omega}^\mu$ will do.

So far, we have only considered restricted Lorentz-transformations, elements in $\mathrm{L}^{\uparrow}_{+}$:

3.5 *For an arbitrary Lorentz-transformation $\varLambda \in \mathrm{L}$, with time and space reflections allowed, one has*

$$\varLambda(\mathrm{W}^{\varepsilon}_{\gamma}) = \mathrm{W}^{\varepsilon\,\mathrm{sgn}\det\varLambda}_{\varLambda.\gamma} \tag{3.13}$$

This is easy to see, because the group-quotients $\mathrm{L}_+/\mathrm{L}_+^\uparrow$ and L/L_+ are both of order two, and since formula (3.13) clearly defines a group action of L on the set of all maximal isotropic vectorspaces $\mathrm{W}_\gamma^\varepsilon$, it is enough to verify the formula for two coset representatives, a transformation Λ_t which reverses the light-cone but with determinant one, and a transformation Λ_r which reverses space-time orientation, that is, the determinant is minus one.

If the vectorspace $\mathrm{W}_\gamma^\varepsilon$, in some restricted orthonormal basis, is of the form $\mathrm{W}_\gamma^\varepsilon = (\omega^0 + \omega^3, \omega^1 + \mathrm{i}\,\varepsilon\omega^2)_\mathbb{C}$, then take as Λ_t the transformation $\omega^0 \to -\omega^0$, $\omega^3 \to -\omega^3$, $\omega^1 \to \omega^1$, $\omega^2 \to \omega^2$; in fact $\Lambda_t(\mathrm{W}_\gamma^\varepsilon) = \mathrm{W}_\gamma^\varepsilon$ (warning: this is an equality of sets, not valid for the elements in the sets), and since $\det \Lambda_t = 1$, formula (3.13) is true in this case. And as Λ_r take $\omega^\mu \to \omega^\mu$ for $\mu \neq 2$, and $\omega^2 \to -\omega^2$; then $\det \Lambda_r = -1$, and Λ_r sends $\mathrm{W}_\gamma^\varepsilon$ to $\mathrm{W}_\gamma^{-\varepsilon}$, so the formula is verified in this case, too. And so formula (3.13) is verified.

Because the reflection Λ_t sends $\mathrm{W}_\gamma^\varepsilon$ to itself, one might be led to believe that it is enough to consider proper Lorentz-transformations, elements of L_+ that have determinant one, but which otherwise need not preserve time-orientation. However, the image of the exponential map is $\mathrm{L}_+^\uparrow$, so elements not in $\mathrm{L}_+^\uparrow$ will have to be studied separately as reflections, and so for this reason, it is important to have the classification being done over the proper Lorentz group. In addition, the elements $dt + dz \in \mathcal{L}^\uparrow$ and $-(dt + dz) \in \mathcal{L}^\downarrow$ have quite different physical interpretation, either as photons moving into the future or past respectively, or (in some quantum mechanical theories) as particle/antiparticles. Because of such reasons, γ will usually be taken to be future-pointing, $\gamma \in \mathcal{L}^\uparrow$.

If we include complex conjugations, we can summarize the following:

3.6 *For each line on the the light-cone $\mathcal{L}$ defined by the element γ, there are exactly two maximal isotropic vectorspaces containing this line, namely W_γ^+ and W_γ^-, and these vectorspaces are the complex conjugates of each other, that is,*

$$(\mathrm{W}_\gamma^\pm)^{\mathrm{cx}} = \mathrm{W}_\gamma^\mp \tag{3.14}$$

This follows from the statements i and ii of proposition 3.2 above, because one writes $\gamma = dt + dz$ in some restricted orthonormal basis of your choice, and then all maximal isotropic vectorspaces W containing γ is of the the form $W = (dt + dz, dx + \mathrm{i}\,\varepsilon dy)_\mathbb{C} =: \mathrm{W}_\gamma^\varepsilon$, and evidently $(\mathrm{W}_\gamma^\varepsilon)^{\mathrm{cx}} = ((dt + dz, dx + \mathrm{i}\,\varepsilon dy)_\mathbb{C})^{\mathrm{cx}} = (dt + dz, dx - \mathrm{i}\,\varepsilon dy)_\mathbb{C} = \mathrm{W}_\gamma^{-\varepsilon}$.

So far we have only studied the classification of single maximal isotropic vectorspaces W; however, if you want to describe the spin representation isomorphism $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) \cong (\frac{1}{2}, \frac{1}{2})$, you need a direct sum decomposition $V_\mathbb{C} \cong W \oplus W'$ of $V_\mathbb{C}$ into two maximal isotropic vectorspaces W and W'. One thing, worthy of note, is that this spin representation isomorphism depends on the choice of a time-like vector, which you may then interpret as a four-momentum in the physical interpretation (the exact interpretation of the presence of this time-like vector is somewhat unclear to me, at the time I write this).

The statement below is by necessity the same as one arrives at using pure spinors over the complex Clifford algebra $\mathrm{C}(g)_{\mathbb{C}}$, since we know, from pure spinor theory, that there are exactly two orbits of the set of maximal isotropic vectorspaces under the action of the proper complex Lorentz group $\mathrm{L}_{\mathbb{C}}^{+}$, and we have shown above that the number of orbits does not become larger under the action of the restricted Lorentz group $\mathrm{L}_{+}^{\uparrow}$.

3.7 Proposition. *Given a pair of* <u>*distinct*</u> *maximal isotropic vectorspaces* $\mathrm{W}_{\gamma}^{\varepsilon}$ *and* $\mathrm{W}_{\check{\gamma}}^{\eta}$*, there are only the following possibilities:*

i. The signs ε *and* η *are equal, so that* γ *and* $\check{\gamma}$ *defines distinct lines on the light-cone* $\mathcal{L}$ *(equivalently* $g(\gamma, \check{\gamma}) \neq 0$*). In this case* $\mathrm{W}_{\gamma}^{\varepsilon} \cap \mathrm{W}_{\check{\gamma}}^{\eta} = 0$ *and* $V_{\mathbb{C}} = \mathrm{W}_{\gamma}^{\varepsilon} \oplus \mathrm{W}_{\check{\gamma}}^{\eta}$*.*

ii. The signs ε *and* η *are unequal (and in this case* $\gamma = \check{\gamma}$ *is allowed). In this case,* $\mathrm{W}_{\gamma}^{\varepsilon} \cap \mathrm{W}_{\check{\gamma}}^{\eta}$ *is of complex dimension one.*

However, working with the Lorentz structure, it is necessary to have a more explicit description, described in the statement below; proposition 3.7 above follows directly from it.

3.8 Proposition. *For a given pair of* <u>*distinct*</u> *maximal isotropic vectorspaces* $\mathrm{W}_{\gamma}^{\varepsilon}$ *and* $\mathrm{W}_{\check{\gamma}}^{\eta}$*, there is a restricted orthonormal basis* ω^{μ} *in which these vectorspaces have the following simultaneous description, according to the following possibilities:*

i. The signs ε *and* η *are equal, so that* γ *and* $\check{\gamma}$ *defines distinct lines on the light-cone* $\mathcal{L}$ *(equivalently* $g(\gamma, \check{\gamma}) \neq 0$*). In this case, one can choose the basis* ω^{μ} *so that the isotropic vectorspaces are equal to* $(dt \pm dz, dx \pm \mathrm{i}\, dy)_{\mathbb{C}}$*.*

ii. The signs ε *and* η *are unequal, and the lines on the light-cone* $\mathcal{L}$ *are equal (equivalently* $g(\gamma, \check{\gamma}) = 0$*). In this case, one can take the basis* ω^{μ} *so that the isotropic vectorspaces are equal to* $(dt + dz, dx \pm \mathrm{i}\, dy)_{\mathbb{C}}$*.*

iii. The signs ε *and* η *are unequal, and the lines on the light-cone* $\mathcal{L}$ *are distinct (equivalently* $g(\gamma, \check{\gamma}) \neq 0$*). In this case, one can take the basis* ω^{μ} *so that the isotropic vectorspaces are equal to* $(dt \pm dz, dx + \mathrm{i}\, dy)_{C}$*.*

Proof: First assume, for simplicity, that both γ and $\check{\gamma}$ are future pointing, otherwise change γ to $-\gamma$ and $\check{\gamma}$ to $-\check{\gamma}$ as needed.

Suppose now that γ and $\check{\gamma}$ define distinct lines on the light-cone $\mathcal{L}$. Choose ω^{μ} by taking ω^{0} to be the time-like future-pointing element $\gamma + \check{\gamma}$ normalized, taking ω^{3} to be the spatial element $\gamma - \check{\gamma}$ normalized, and extending arbitrarily to a restricted orthonormal basis. Then $dt + dz$ is equal to γ up to a scalar, and $dt - dz$ is equal to $\check{\gamma}$ up to a scalar.

By statement ii of proposition 3.2 above, the only maximal isotropic vectorspaces through $dt + dz$ and $dt - dz$ are the vectorspaces $(dt + dz, dx + \mathrm{i}\,\varepsilon dy)_{\mathbb{C}}$ and $(dt - dz, dx - \mathrm{i}\,\eta dy)_{\mathbb{C}}$, for some signs $\varepsilon, \eta \in \{+1, -1\}$. In addition, if (dt, dx, dy, dz) is a restricted orthonormal basis, so is $(dt, dx, -dy, -dz)$, and it follows from this that these maximal isotropic vectorspaces have the same sign exactly when $\varepsilon = \eta$, and opposite signs if $\varepsilon \neq \eta$. This, in fact, shows statements i and iii of the proposition.

If, on the other hand, γ and $\check{\gamma}$ define the same line on the light-cone $\mathcal{L}$, it follows from statements i and ii of proposition 3.2 that there is a restricted orthonormal basis ω^μ, in which the maximal isotropic vectorspaces are of the form $(dt + dz, dx \pm \mathrm{i}\, dy)_{\mathbb{C}}$. Since the vectorspaces are distinct, this shows statement ii of the proposition.

4 Tensor, Exterior and Clifford Algebras

So far only the coordinate versions of the spin operators $\sigma^j_\pm$ and the maximal isotropic vectorspaces $\mathrm{W}^\pm_\gamma$ have been developed. For several reasons, it will be necessary to proceed with the coordinatefree versions. One such reason, as was indicated above in section 2, is to ensure that the operators $\sigma^j_\pm$ (or, rather their linear span over $\mathbb{R}$) exist independently of the coordinate system chosen, and also that the fact that they leave the maximal isotropic vectorspaces $\mathrm{W}^\pm_\gamma$ invariant is not coordinate system dependent.

Before this can take place, we have to spend some time defining the tools used, that is, the tensor and exterior algebras, in part because there are different conventions in use. In addition, even though it is the case that the standard Dirac spin representation of the Clifford bundle (or algebra, if you are at the stalk) $\mathrm{C}(g)$ does not lead to the spin theory presented here, it turns out that the Clifford algebra is the most convenient way to describe the tensor symmetries at hand. In part this is so, because the the Hodge star operator $*$ corresponds, up to a sign, with Clifford multiplication by the volume element ω, and it is possible to use this latter element interchangibly as an operator or an element.

In addition to that, even though it is the case that the standard Dirac spin representation of the Clifford bundle $\mathrm{C}(g)$ does not lead to the spin theory presented here, it will later turn out that the spin representation isomorphism $(\frac{1}{2},0)\otimes(0,\frac{1}{2})\cong(\frac{1}{2},\frac{1}{2})$ is conveniently described using the even part $\mathrm{C}(g)^{\mathrm{even}}$ of the Clifford bundle.

One can also descibe the spin representations of type $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, represented by the spin-operators $\sigma^j_\pm$ acting on the maximal isotropic vectorspaces $\mathrm{W}^\pm_\gamma$, by having the algebra $\mathrm{C}(g)^{\mathrm{even}}$ acting on $\mathrm{C}(g)^{\mathrm{odd}}$, the odd part of the Clifford algebra. In such a description, the vectorspaces $\mathrm{W}^\pm_\gamma$ will be of mixed degree in $\mathrm{C}(g)^{\mathrm{odd}}$, that is, consisting of both degree one and three elements. In this description, the projection operators $\frac{1}{2}(1\pm \mathrm{i}\,\omega)$ are involved in a certain way; it will be described below. This description has the advantage of extricating the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ type spin representations in a more natural way, rather than the brute force method of simply truncating one of the sets of operators $\sigma^j_\pm$ to zero.

It should be remarked that the descriptions above, using the algebra $\mathrm{C}(g)^{\mathrm{even}}$, do not extend, using the canonical embeddings of $\mathrm{C}(g)^{\mathrm{even}}$ and $\mathrm{C}(g)^{\mathrm{odd}}$ into $\mathrm{C}(g)$, to the algebra $\mathrm{C}(g)$ — the vectorspaces invariant under $\mathrm{C}(g)^{\mathrm{even}}$ are being flipped out of place by $\mathrm{C}(g)$. This is not a big problem if you really need the Dirac equation, since once one have a spin theory, one can always resurrect the Dirac equation using so called Infeld-van der Waerden symbols[35], if one so wishes.

We will return to these topics later. So let us move from this survey to the specifics then. The detailed description below is needed, so one does not put

[35]See Penrose-Rindler [15], vol 1, equ (3.1.48), p 124 and footnote same page.

in erroneous signs, or misplaced factors like 2, which could lead to that spin and antispin is being confused, or that one instead of the spin-representation arrives with the orthogonal representation after an exponentiation. However, the descriptions below are mainly intended as a fixing of notation, and not as an introduction to differential geometry.

4.1 Coordinates and Sheaves

Even though it is perfectly possible to use cotensors, the elements of the cotangent bundle $\mathrm{T}^*M =: \mathrm{T}^{0,1}M$, as a starting point, defined as local infinitesimal approximations (that is, at each point a of the smooth manifold M, as elements of m_a/m_a^2, where $m_a := \{f \in \mathcal{C}_a^\infty M \mid f(a) = 0\}$ is the local maximal ideal in the ring of function germs $\mathcal{C}_a^\infty M$ at the point a, and smoothly varying to neighbouring points), the common way to go is by first defining the set of vectorfields with smooth coefficients $\mathrm{T}M =: \mathrm{T}^{1,0}M$, called the tangent bundle then, as the set of derivations $X\colon \mathcal{C}^\infty M \to \mathcal{C}^\infty M$, that is, satisfying $X(f_1 f_2) = X(f_1)f_2 + f_1 X(f_2)$ for any smooth functions $f_1(x), f_2(x) \in \mathcal{C}^\infty M$.

At each point a of the smooth manifold M the set of tangent vectors M_a is by definition the set of $\mathbb{R}$ linear derivations $v\colon \mathcal{C}^\infty M \to \mathbb{R}$, that is $v(\alpha f_1 + \beta f_2) = \alpha v(f_1) + \beta v(f_2)$ and $v(f_1 f_2) = v(f_1)f_2(a) + f_1(a)v(f_2)$, for scalars $\alpha, \beta \in \mathbb{R}$ and functions $f_1(x), f_2(x) \in \mathcal{C}^\infty M$. If one has, at each point a of the smooth manifold M, a tangent vector v_a, and these tangent vectors are smoothly varying in the sense that for any smooth function $f_1 \in \mathcal{C}^\infty M$, the function $f_2(a) := v_a(f_1)$ is also smooth, then a corresponding vectorfield X can be defined from this smoothly varying family of tangent vectors v_a by the formula $X(f_1) := f_2$, and conversely, for any smooth vectorfield X, this formula also defines a smoothly varying family of tangent vectors v_a.

Then the set of cotensors T^*M is defined as the $\mathcal{C}^\infty M$ dual of the module of vectorfields $\mathrm{T}M$, in other words, an element $\alpha \in \mathrm{T}^*M$ is a linear function $\alpha\colon \mathrm{T}M \to \mathcal{C}^\infty M$ satisfying $\alpha(f_1 X + f_2 Y) = f_1\alpha(X) + f_2\alpha(Y)$ for smooth functions $f_1, f_2 \in \mathcal{C}^\infty M$ and vectorfields $X, Y \in \mathrm{T}M$.

Similarly, at a point a of the smooth manifold M, the set of cotangent vectors M_a^* is defined as the $\mathbb{R}$ linear dual of the vectorspace of tangent vectors M_a, so, in other words, it consists of $\mathbb{R}$ linear maps $w\colon M_a \to \mathbb{R}$. However, as pointed out above, one can identify $M_a^* = m_a/m_a^2$, where $m_a := \{f \in \mathcal{C}_a^\infty M \mid f(a) = 0\}$ is the local maximal ideal in the ring of smooth function germs at a, because there is a non-degerate bilinear pairing $M_a \times m_a/m_a^2 \to \mathbb{R}$, sending a pair (v, f) to $v(f)$, and which corresponds to a Taylor approximation of f to the degree one.

Working this out in coordinates, if e_μ is a local coordinate basis at the point a of the smooth manifold M, with the corresponding coordinate functions x^μ which goes from a neighbourhood of a in M to $\mathbb{R}$, a basis for M_a is the set of tangent vectors $\frac{\partial}{\partial x^\mu}\big|_{x=a}$, and letting such tangent vectors move with a, which they do smoothly since the coordinate functions are always assumed to be smooth, shows that the vectorfields locally are of the form $X = X^\mu \frac{\partial}{\partial x^\mu}$, where $X^\mu = X^\mu(x)$ are smooth functions.

The cotensors $\alpha \in \mathrm{T}^*M$ can then be written on the form $\alpha = \alpha_\mu dx^\mu$, where again the $\alpha_\mu = \alpha_\mu(x)$ are smooth functions, and the (dx^μ) can be described as either the dual basis of the basis $(\frac{\partial}{\partial x^\nu})$ of $\mathrm{T}M$, that is $dx^\mu(\frac{\partial}{\partial x^\nu}) = \delta^\mu{}_\nu$, the Kronecker delta, or alternatively as the differentials of the coordinate functions $x^\mu(x)$ (details are omitted).

One should note that this description with a local basis only works because the manifold M is smooth, and the coordinate functions thus also are smooth. For non-smooth manifolds, like manifolds with singularities, the right thing is to invoke sheaf theory, even though I would expect problems with singularities to be resolved in a quantum wave cutoff — this would happen roughly because one would expect that the potentials that causes singularity phenomena to occur, themselves be created by statistical waves in a suitable quantum wave correction. From this point of view, the study of singularities will appear as a first approximation of the complete picture.

A mathematical phenomenon is called *infinitesimal* if it takes place at the stalk, it is called *local* if it takes place in a neighbourhood, and it is called *global* if it takes place in neighbourhoods and admits being glued together in a suitable sense. In sheaf theory such otherwise (especially in physics) only intuitive concepts become precise.

Sheaves are also useful in order to pinpoint the data of the theory that you are using. For example, let $\mathcal{C}^\infty_U$ denote the sheaf on the open set $U \subset \mathbb{R}^n$ that to the open subset $U_0 \subset U$ assigns the set of smooth functions $\mathcal{C}^\infty(U_0)$ on that set; the stalk $\mathcal{C}^\infty_a$ at any point $a \in U$ then just consists of the function germs at this point. An n dimensional smooth manifold M may then be identified with a sheaf that is locally isomorphic to the family of sheaves $\mathcal{F} := (\mathcal{C}^\infty_U \mid U$ is open subset of $R^n)$; the family of sheaves $\mathcal{F}$ is called a model for the family of smooth manifolds.

Using this language then, an extrinsic spin theory à la Penrose-Rindler [15] (also confer Benn & Tucker [2], ch 9, and Wells [19]) may then be described as follows (details omitted): One takes as models, for open subsets U of the Minkowski space $\mathcal{M}$, the sheaves $\mathcal{C}^\infty_U(S)$, which to an open subset $U_0 \subset U$ assigns the set of smooth functions $f(x)$ on U_0 with values in a pre-chosen spin bundle representation S of the Clifford bundle $\mathrm{C}(g)$ on $\mathcal{M}$. Since one has global coordinates in the Minkowski space $\mathcal{M}$ to rely on, there is no problem to predefine such a global spin representation bundle on the Minkowski space.

If you are using the corresponding version of this construction on a compact manifold in a positive definite metric, and is only interested in global topological or cohomological data, the details of how this spin representation bundle is defined locally is of subordinate importance, since globally equivalent bundles will yield the same global invariants. However, in order to be able to compute such things as the generalization of the Einstein equation and other data relating to quantum mechanics, such local data has to be specified, which is why we end up having to construct a mathematically intrinsic spin theory.

Continuing the description of the tensors then, once we have constructed the vectorfield vectorspace $\mathrm{T}M$ and the cotensor vectorspace T^*M, the tensor space $\mathrm{T}^{p,q}M$ of tensors of type, or bidegree, (p,q) can be defined as

$$\mathrm{T}^{p,q}M := \underbrace{\mathrm{T}M \otimes \ldots \otimes \mathrm{T}M}_{p \text{ times}} \otimes \underbrace{\mathrm{T}^*M \otimes \ldots \otimes \mathrm{T}^*M}_{q \text{ times}} \tag{4.1}$$

that is, $\mathrm{T}^{p,q}M$ consists of elements that are sums of elements of the form $X_1 \otimes \ldots \otimes X_p \otimes \alpha_1 \otimes \ldots \otimes \alpha_q$, where $X_1, \ldots, X_p \in \mathrm{T}M$ are vectorfields and the $\alpha_1, \ldots, \alpha_q \in \mathrm{T}^*M$ are one degree cotensors. Note that by this, $\mathrm{T}^{1,0}M = \mathrm{T}M$ and $\mathrm{T}^{0,1}M = \mathrm{T}^*M$.

In local coordinates, the elements A in $\mathrm{T}^{p,q}M$ are of the form

$$A = \sum_{\substack{\mu_1 \cdots \mu_p \\ \nu_1 \cdots \nu_p}} A^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} \, \tfrac{\partial}{\partial x^{\mu_1}} \otimes \ldots \otimes \tfrac{\partial}{\partial x^{\mu_p}} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_q} \tag{4.2}$$

where the components $A^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} \in \mathcal{C}^\infty M$ are smooth coordinate functions. Note that the summation sign is also most often dropped, and in order to arrive at classical tensor notation, one also drops the tensorbasis $\frac{\partial}{\partial x^{\mu_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\mu_p}} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_q}$, so that one represents A only by the tensor components $A^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$, which is convenient in explicit computations.

Alternatively, the elements of $\mathrm{T}^{p,q}M$ may be described as the set of $\mathcal{C}^\infty M$ mulitilinear maps

$$\underbrace{\mathrm{T}^*M \times \ldots \times \mathrm{T}^*M}_{p \text{ times}} \times \underbrace{\mathrm{T}M \times \ldots \times \mathrm{T}M}_{q \text{ times}} \longrightarrow \mathcal{C}^\infty M \tag{4.3}$$

In fact, this amounts only to say that the dual of $\mathrm{T}^{q,p}M$ over the ring $\mathcal{C}^\infty M$ is isomorphic to $\mathrm{T}^{p,q}M$. Since $\mathrm{T}M$ and T^*M are duals of each other, we can quite freely dualize like this, but especially in the case of symmetric and antisymmetric tensors, one has to watch out for dualizing coefficients — in fact, one reason for this tensor review, is to make sure that no such unwanted coefficients sneak into the picture. We will return to these dualizing pairings below.

Note that, if $X \in \mathrm{T}M$ is a vectorfield and $\alpha \in \mathrm{T}^*M$ is a cotensor, the tensors $X \otimes \alpha$ and $\alpha \otimes X$ will both be described as tensors of type $(1,1)$, and they will not be identified, because for instance when dealing with metric conversions of two-forms, such an identification would introduce unwanted minus signs. A more appropriate indexing than a number pair (p,q) would then be a map between finite sets $\{1, ..., p+q\} \to \{\text{vectorfield, cotensor}\}$, but the common approach to resolve this difficulty is to forget about the problem altogether.

4.2 Distribution Coefficient Tensors

So far we have only discussed tensors with smooth coefficient functions. In order to be able to discuss topics like unification of point particles models and particle flow models, and also in order to be able to do analysis, particularly in the context of Sobolev spaces, one needs to employ tensors with distribution coefficients. It is not difficult to define tensors with distribution coefficients, but as it is only needed marginally in this paper, we skip over this topic altogether.

Considered as bundles, the tangent bundle $\mathrm{T}M$ and the cotangent bundle T^*M comes with canonical projections $\mathrm{T}M \to M$, $\mathrm{T}^*M \to M$. The tangent bundle $\mathrm{T}M$ and the cotangent bundle T^*M can also be considered as manifolds, of dimension twice the dimension of the manifold M, which is eight then if M is a fourdimensional Lorentz manifold. In physics, T^*M corresponds to considering all the kets $|x, k\rangle$ simultanesously, where x is a spatial point and $k = k_\mu = k_\mu dx^\mu$ is the dual of the energy-momentum.

One can then add a relation between x and k, like for instance the kernel $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}k\cdot x}$, in which case results in what physicists call an operator normal ordering, and what mathematicians call (in the case of a positive definite metric) pseudodifferential operators. Such a theory can the be enhanced somewhat to include a theory of singularities, using so called D-modules (differential modules), which, in their turn, are naturally equivalent to intersection cohomology.

The analysis of the manifold T^*M should not be confused with the analysis of the manifold M, but for the scope of this paper, we need not dwell upon this subject.

4.3 Differential Forms and Antisymmetric Tensors

We continue with the definition of the algebra of differential forms $\Omega^{\cdot}M$. As was indicated above, the purpose of this review is mainly to fix notation, and indicate especially how the transition between the more abstract theory and the explicit tensor notation is done. The details of this theory can be found in books on differential geometry, like for instance Warner [18], ch 2, and Helgason [7].

Below the dualizing is generally made as $\mathcal{C}^\infty M$ modules. One other approach, which gives the same results, is to go down on the stalks of the objects where everything consists of $\mathbb{R}$ vector spaces, that is, for a point $a \in M$, one replaces $\mathrm{T}M$ by M_a and T^*M by M_a^*, and so forth. One so performs all algebraic operations locally, and after one has done this, one lets the objects move smoothly with the point a on the manifold M again. Such an approach is the correct one if one is dealing with sheaf theory (the category of sheaves is deliberately constructed in such a way that maps of sheaves corresponds to local maps glued together in the described fashion), but in the smooth theory, because one can always bend smooth functions locally, it is enough to consider $\mathcal{C}^\infty M$ as a ring instead as a sheaf of rings. One advantage of the approach here, is that it is somewhat more explicit.

A differential form $\alpha \in \Omega^p M$, of degree p then, is commonly defined as an alternating function

$$\alpha\colon \underbrace{\mathrm{T}M \times \ldots \times \mathrm{T}M}_{p \text{ times}} \longrightarrow \mathcal{C}^\infty M \tag{4.4}$$

multilinear over $\mathcal{C}^\infty M$ then, and satisfying $\alpha\left(X_{\sigma(1)}, \ldots, X_{\sigma(p)}\right) = \operatorname{sgn}\sigma\ \alpha\left(X_1, \ldots, X_p\right)$ for vectorfields $X_1, \ldots, X_p$, and for a permutation σ of the numbers $(1, \ldots, p)$. Here, $\Omega^p M$ is really shorthand notation for $\Omega^p(\mathrm{T}M)$, confer the section on notation.

Alternatively, one can take as definition for a differential form β of degree p to be that β is an element of the exterior algebra $\bigwedge^p(\mathrm{T}^*M)$, which can be constructed as the algebra $\mathrm{T}(\mathrm{T}^*M)/(x \otimes x = 0 \mid x \in \mathrm{T}^*M)$, that is, one takes the total tensor algebra of the $\mathcal{C}^\infty M$ module T^*M and divides out with the ideal consisting of the squares of the degree one tensors; note that $\mathrm{T}(\mathrm{T}^*M) = \mathrm{T}^{0,\cdot}M := \bigoplus_{q=0}^\infty \mathrm{T}^{0,q}M$.

An exterior algebra can be described as the algebra that solves a universal problem with respect to alternating multilinear functions. Because of this universal property (details omitted), the exterior algebra dual $(\bigwedge^p(\mathrm{T}M))^*$, dualized over $\mathcal{C}^\infty M$ then, and $\Omega^p(TM)$ are isomorphic, which can also be seen as the result of the existence of the non-degenerate $\mathcal{C}^\infty M$ bilinear pairing

$$\begin{cases} \Omega^p(TM) \times \bigwedge^p(\mathrm{T}M) \longrightarrow \mathcal{C}^\infty M \\ \quad f \times X_1 \wedge \ldots \wedge X_p \longrightarrow f\left(X_1, \ldots, X_p\right) \end{cases} \tag{4.5}$$

The exterior algebras $\bigwedge^p(\mathrm{T}^*M)$ and $(\bigwedge^p(\mathrm{T}M))^*$ are also isomorphic, via the bilinear pairing

$$\begin{cases} \qquad\qquad \bigwedge^p(\mathrm{T}^*M) \times \bigwedge^p(\mathrm{T}M) \longrightarrow \mathcal{C}^\infty M \\ \alpha_1 \wedge \ldots \wedge \alpha_p \times X_1 \wedge \ldots \wedge X_p \longrightarrow \det(\alpha_j(X_k))_{j,k=1,\ldots,p} \end{cases} \tag{4.6}$$

It is in fact the different possible choices of this pairing that causes constant confusion, that is, the constants of the operations on tensors in $\Omega^\cdot M$ may be subject to confusion, because some people want to put in a factor $\frac{1}{p!}$ in front of the determinant in the above formula, perhaps because being influenced by the subject of symmetrization, confer Warner [18], 2.10b, p60 for details on this matter. The approach adopted here leads to the conventions most physicists use, and is thus convenient both from the practical and the theoretical point of view.

Nevertheless, the isomorphisms

$$\Omega^p M \cong (\textstyle\bigwedge^p(\mathrm{T}M))^* \cong \bigwedge^p(\mathrm{T}^*M) \tag{4.7}$$

makes the exterior multiplication $\wedge$ on $\bigwedge^\cdot M$ induce a multiplication on $\Omega^\cdot M$, and which is also denoted by $\wedge$.

One can see that, for $\alpha \in \Omega^p M$ and $\beta \in \Omega^q M$, this multiplication is

$$\alpha \wedge \beta\,(X_1, \dots, X_{p+q}) = \sum_{\substack{\sigma \\ (p,q)\text{ shuffle}}} \operatorname{sgn}\sigma\;\; \alpha\big(X_{\sigma(1)}, \dots, X_{\sigma(p)}\big)\, \beta\big(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}\big) \tag{4.8}$$

where $X_1, \dots, X_{p+q}$ are vectorfields, σ is a permutation of $(1, \dots, p+q)$, and where σ is called a (p,q) shuffle if it is increasing on the two intervals $[1,p]$ and $[p+1, p+q]$, that is, $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$. (The (p,q) shuffle is named so because it corresponds to the operations gotten by splitting a card deck with $p+q$ cards into two parts with p respectively q elements, and then shuffle it.)

For example, if α is a one-form and β is a two-form, then

$$(\alpha \wedge \beta)(X_1, X_2, X_3) = \alpha(X_1)\beta(X_2, X_3) + \alpha(X_2)\beta(X_3, X_1) + \alpha(X_3)\beta(X_1, X_2) \tag{4.9}$$

for vectorfields X_1, X_2, X_3.

It is time to work this out in coordinates. The above isomorphism $\bigwedge^{\cdot}(\mathrm{T}^*M) \cong (\bigwedge^{\cdot}(\mathrm{T}M))^*$ also induces an identification of $\bigwedge^{\cdot}(\mathrm{T}^*M)$ with a subset of $\mathrm{T}(\mathrm{T}^*M) = \mathrm{T}^{0,\cdot}M$, because the projection $\mathrm{T}(\mathrm{T}M) \to \bigwedge(\mathrm{T}M)$ dualizes to an injection $(\bigwedge(\mathrm{T}M))^* \to (\mathrm{T}(\mathrm{T}M))^*$, so one gets

$$\textstyle\bigwedge^{\cdot}(\mathrm{T}^*M) \cong (\bigwedge^{\cdot}(\mathrm{T}M))^* \xrightarrow{\subset} (\mathrm{T}(\mathrm{T}M))^* \cong \mathrm{T}(\mathrm{T}^*M) = \mathrm{T}^{0,\cdot}M \tag{4.10}$$

One can see that for the one-forms $\alpha_1, \dots, \alpha_p \in \Omega^1 M = \mathrm{T}^*M$, the above identification gives

$$\alpha_1 \wedge \dots \wedge \alpha_p = \sum_{\sigma} \operatorname{sgn}\sigma\;\; \alpha_{\sigma(1)} \otimes \dots \otimes \alpha_{\sigma(p)} \tag{4.11}$$

where σ is a permutation of $(1, \dots, p)$. For instance, for one-forms α, β, one gets $\alpha \wedge \beta = (\alpha \otimes \beta - \beta \otimes \alpha)$.

If now $A = A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \otimes \dots \otimes dx^{\mu_p}$ is an antisymmetric tensor, meaning that the interchange of any two indices of $A_{\mu_1 \dots \mu_p}$ changes the sign of this coefficient, then applying above formula (4.11) above to the one-forms $dx^1, \dots, dx^p$ yields the following formula:

$$\begin{aligned} A &= A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \otimes \dots \otimes dx^{\mu_p} \\ &= \sum_{\mu_1 < \dots < \mu_p} A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \\ &= \frac{1}{p!} \sum_{\mu_1, \dots, \mu_p} A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \end{aligned} \tag{4.12}$$

This is the formula to use when converting between the more structured mathematical notions, and the tensor component notation appearing in physics literature. For example, setting $c = 1$ so that $dx^0 = c\,dt = dt$, if $F = E_x dt \wedge dx = E_x(dt \otimes dx - dx \otimes dt)$, then $F_{01} = E_x$ and $F_{10} = -E_x$, and so on.

4.4 Symmetric Tensors

In addition to the antisymmetric tensors, one can make use of the symmetric tensor algebra $\mathrm{S}(\mathrm{T}^*M)$. This algebra shows up in two contexts, one being that it describes higher differentials, in the sense of higher order infinitesimal Taylor approximations, confer Warner [18], §1.26, p 20. The other context is that the irreducible representations of the spin Lie algebra (which is the same as the orthogonal Lie algebra $\mathrm{o}(g)$, where g is a Lorentz metric, or alternatively, one can in a time evolution picture use $\mathrm{o}(3)$) can be constructed as symmetric tensors — we will leave this latter topic for now, but note that the images of the earlier constructed spin representations $\mathrm{W}_\gamma^\pm$ inside T^*M (or rather, at the stalk M_a^* at some point $a \in M$) will generate the higher irreducible spin representations.

The symmetric tensor algebra $\mathrm{S}(\mathrm{T}M)$ can be defined as $\mathrm{T}(\mathrm{T}M)/(X \otimes Y = Y \otimes X \mid X, Y \in \mathrm{T}M)$, that is, on the total tensor algebra $\mathrm{T}(\mathrm{T}M)$, you divide out by all the ideal generated by all commutators $[X, Y] = X \otimes Y - Y \otimes X$, where X and Y are degree one tensors (vectorfields). The symmetric tensor algebra $\mathrm{S}(\mathrm{T}M)$ also solves a universal problem with respect to multilinear symmetric functions $f\colon\ \mathrm{T}M \times \ldots \times \mathrm{T}M \longrightarrow \mathcal{C}^\infty M$. Similar statements are true about the symmetric tensor algebra $\mathrm{S}(\mathrm{T}^*M)$. of course.

The formulas related the symmetric tensor algebra $\mathrm{S}(\mathrm{T}^*M)$ are the same as those related to the exterior algebra $\bigwedge^{\cdot}(T^*M)$, except that throughout one drops the sign $\operatorname{sgn}\sigma$ of the permutations σ involved in those formulas.

Thus, there is an isomorphism $\mathrm{S}^p(\mathrm{T}^*M) \cong (\mathrm{S}^p(\mathrm{T}M))^*$, defined by the non-degenerate bilinear pairing

$$(4.13)\qquad \begin{cases} \mathrm{S}^p(\mathrm{T}^*M) \times \mathrm{S}^p(\mathrm{T}M) \longrightarrow \mathcal{C}^\infty M \\ \alpha_1 \odot \ldots \odot \alpha_p \times X_1 \odot \ldots \odot X_p \longrightarrow \operatorname{sym}(\alpha_j(X_k))_{j,k=1,\ldots,p} \end{cases}$$

where $\odot$ denotes symmetric multiplication, and where sym is an introduced notion of a *symmetrant*, which for a rank p square matrix $(A_{jk})_{j,k=1,\ldots,p}$ is defined to be

$$(4.14)\qquad \operatorname{sym} A := \sum_\sigma \prod_{n=1}^p A_{1\sigma(1)} \cdot \ldots \cdot A_{p\sigma(p)}$$

where σ is a permutation of $(1, \ldots, p)$.

This formula should compared with the corresponding one for the determinant, which is

$$(4.15)\qquad \det A := \sum_\sigma \operatorname{sgn}\sigma \prod_{n=1}^p A_{1\sigma(1)} \cdot \ldots \cdot A_{p\sigma(p)}$$

For generalities on this subject, see Bourbaki *Algebra* [4], ch III, §11.9, formulas (66)–(68), p 604.

Similar to the case of exterior algebras, the isomorphism $(\mathrm{S}(\mathrm{T}M))^* \cong \mathrm{S}(\mathrm{T}^*M)$ induces a multiplication on multilinear symmetric functions $\alpha\,(X_1, \dots, X_p) \in (\mathrm{S}^p(\mathrm{T}M))^*$ and $\beta\,(X_1, \dots, X_q) \in (\mathrm{S}^q(\mathrm{T}M))^*$, which can be seen to be

$$\alpha \odot \beta\,(X_1, \dots, X_{p+q}) = \sum_{\substack{\sigma \\ (p,q)\ \text{shuffle}}} \alpha\big(X_{\sigma(1)}, \dots, X_{\sigma(p)}\big)\, \beta\big(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}\big) \tag{4.16}$$

where, as above, a permutation σ of $(1, \dots, p+q)$ is called a (p, q) shuffle if it is increasing on the intervals $[1, p]$ and $[p+1, p+q]$.

The same isomorphism $(\mathrm{S}(\mathrm{T}M))^* \cong \mathrm{S}(\mathrm{T}^*M)$ can be used to embed $\mathrm{S}(\mathrm{T}^*M) \xrightarrow{\subset} \mathrm{T}(\mathrm{T}^*M) = \mathrm{T}^{0,\cdot}M$, just as in the case of the exterior tensor algebra, and for degree one cotensors $\alpha_1, \dots, \alpha_p \in T^*M$, one can see that it results in the following formula:

$$\alpha_1 \odot \ldots \odot \alpha_p = \sum_{\sigma} \alpha_{\sigma(1)} \otimes \ldots \otimes \alpha_{\sigma(p)} \tag{4.17}$$

So, for instance, for $\alpha, \beta \in \mathrm{T}^*M$, one has $\alpha \odot \beta = \alpha \otimes \beta + \beta \otimes \alpha$.

This can then be applied, to work it out in coordinates: If $A = A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p}$ is a symmetric tensor, meaning that the interchange of any two indices of $A_{\mu_1 \dots \mu_p}$ leaves this coefficient unchanged in value, then applying the above formula to the one-forms $dx^1, \dots, dx^p$ yields the following formula:

$$\begin{aligned} A &= A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} \\ &= \frac{1}{c_{\mu_1 \dots \mu_p}} \sum_{\mu_1 \le \dots \le \mu_p} A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \odot \ldots \odot dx^{\mu_p} \\ &= \frac{1}{p!} \sum_{\mu_1, \dots, \mu_p} A_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \odot \ldots \odot dx^{\mu_p} \end{aligned} \tag{4.18}$$

where the coefficients $c_{\mu_1 \dots \mu_p} := k_1! \dots k_r!$, if the indices $\mu_1, \dots, \mu_p$ fall into r different groups of equal indices, with the number of members $k_1, \dots, k_r$ in these different groups.

Similar to formula (4.12) in case of antisymmetric tensors, this is the formula to use when converting between the more structured mathematical notions, and the tensor component notation appearing in physics literature.

4.5 The Clifford Algebra

In addition to the above tensor algebras, we shall need to make use of the Clifford algebra $\mathrm{C}(g)$ as well. An easy access reference on the subject is Benn &

Tucker [2], ch 2; less easy access references include Chevalley [5] and Bourbaki *Algebra* [4], ch IX, §9.

Even though the spin theory worked out in sections 2 and 3 in terms of coordinates, can be described using only the Hodge star operator, the Clifford algebra $\mathrm{C}(g)$ turns out to be more convenient. In part this is because the square of the volume element $\omega \in \mathrm{C}(g)$ (as opposed to the Hodge star operator $*$) is always minus one, also when acting on tensors of degree one and three, and in part this is so because the volume element ω, and other elements of $\mathrm{C}(g)$, can be considered both as elements and as operators, the latter via Clifford multiplication. In addition, it turns out that the Clifford algebra (or rather its even part $\mathrm{C}(g)^{\mathrm{even}}$ acting on its odd part $\mathrm{C}(g)^{\mathrm{odd}}$) is a convenient way to describe the spin isomorphism $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) \cong (\frac{1}{2}, \frac{1}{2})$. One can also mention that this latter isomorphism in the mathematically intrinsic spin theory presented here depends on the choice of a time coordinate.

Before starting with the actual description of the Clifford algebra $\mathrm{C}(g)$, note that, as above, one can either work locally at the stalk, starting off with the cotangent vectorspace M_a^* at some point $a \in M$, or one can work globally, using the cotensor space $\mathrm{T}^* M$. In the first case one ends up with a Clifford algebra that strictly speaking should be denoted by $\mathrm{C}(g)_a$, with the index a indicating that one is actually working at the stalk at this point of M. The corresponding $\mathrm{C}(g)$, which may then more appropriately called the Clifford bundle, then corresponds of tensors with coefficients being smooth functions with values in the infinitesimal copies $\mathrm{C}(g)_a$. But because this relation infinitesimal/global is so standard, sometimes this index a is being omitted — the context will tell.

There are several ways to describe the Clifford algebra $\mathrm{C}(g)_a$. One way is to say that it is universal with respect to pairs $(f, \mathcal{A})$, where $\mathcal{A}$ is an algebra over $\mathbb{R}$, and f is a linear map $f\colon M_a^* \to \mathcal{A}$ satisfying $f(x)^2 = q(x)$, for $x \in M_a^*$ (or, equivalently, the anticommutator $\{f(x), f(y)\} = 2g(x, y)$, for $x, y \in M_a^*$). This amounts to saying that $\mathrm{C}(g)_a \cong \mathrm{T}(M_a^*)/(\alpha \otimes \alpha = q(\alpha) \mid \alpha \in M_a^*)$, that is, on the tensor algebra $\mathrm{T}(M_a^*)$ you divide out with the ideal generated by elements of the form $\alpha \otimes \alpha - q(\alpha)$. The linear map $f\colon M_a^* \to \mathrm{C}(g)_a$ is simply the composite $f\colon M_a^* \to T(M_a^*) \to \mathrm{C}(g)_a$, and one can see that it is an injection. Since this ideal that you divide out with, consists solely of even degree tensors, $\mathrm{C}(g)_a$ becomes a $\mathbb{Z}_2$ graded algebra (odd and even degrees).

Moreover, the filtration of degrees $\mathrm{F}^p \mathrm{T} M_a^* := \{\alpha \in \mathrm{T} M_a^* \mid \deg \alpha \leq p\}$, induces a filtration $\mathrm{F}^p \mathrm{C}(g)_a$ which is the vectorspace generated (over $\mathbb{R}$) by the elements of $\mathrm{C}(g)_a$ that can be written as a product of at most p elements in $M_a^* \subset \mathrm{C}(g)_a$. One can see that the corresponding graded algebra $\mathrm{gr}\, \mathrm{C}(g)_a$ may be identified with the exterior algebra $\bigwedge M_a^* = \Omega^{\cdot}_a$.

Without going into details, one way to see this is to, hands on, take an orthonormal basis $\omega^\mu := dx^\mu$ and have a look at the elements $\omega^{\mu_1} \circ \ldots \circ \omega^{\mu_p}$, $\mu_1 < \ldots < \mu_p$, for $p = 1, \ldots, \dim M$, which turns out to be a basis of $\mathrm{C}(g)_a$ considered as a vectorspace over $\mathbb{R}$. Then $\mathrm{F}^k \mathrm{C}(g)_a$ is generated, as a vectorspace over $\mathbb{R}$, by the basis elements $\omega^{\mu_1} \circ \ldots \circ \omega^{\mu_p}$ for $p \leq k$, and passing to $\mathrm{gr}\mathrm{C}(g)_a =$

$\oplus \mathrm{F}^k/\mathrm{F}^{k-1}$, one sees that the images of those basis elements behave as if they were in the exterior algebra $\bigwedge(M_a^*) = \Omega_a^{\cdot} M$.

These facts corresponds to computations done by physicists in terms of the γ^μ matrices of the Dirac equation (the linear spans are then over $\mathbb{C}$). But the γ^μ matrices correspond to an isomorphism $(\mathrm{C}(g)_a)_{\mathbb{C}} \cong \mathrm{End}(S)$, where S is a choice of a (Dirac) spin representation, so one becomes completely dependent on the choice of this spin representation, and, among other things, it is not possible to compute the metric variations. So we have to proceed differently.

Also, without going into details for now, one can set up an isomorphism $\Phi_a \colon \Omega_a^{\cdot} M \xrightarrow{\cong} \mathrm{C}(g)_a$, with $\Omega_a^{\cdot} M$ and $\mathrm{C}(g)_a$ being considered as real vectorspaces, by sending $\omega^{\mu_1} \wedge \ldots \wedge \omega^{\mu_p}$ to $\omega^{\mu_1} \diamond \ldots \diamond \omega^{\mu_p}$, keeping the notation from above. One can see, by relying on a more abstract description, that this isomorphism does not depend on the choice of basis.

Moreover, $\Omega_a^{\cdot} M$ can be made into a left $\mathrm{C}(g)_a$ module as follows: For $x \in M_a^*$ and $\alpha \in \Omega_a^{\cdot} M$, let $e(x)\alpha := x \wedge \alpha$ denote exterior multiplication and let $i(x)\alpha$ denote interior multiplication, defined as the metric dual of $e(x)$, that is, $g(i(x)\alpha, \beta) :=: g(\alpha, e(x)\beta)$. Then $x \in M_a^*$ is made acting on $\alpha \in \Omega_a^{\cdot} M$ by the formula $c(x).\alpha := (e(x) + i(x)).\alpha$; one can see that $c(x)^2 = q(x)$, so the formula extends to a Clifford multiplication, of $\mathrm{C}(g)_a$ acting on $\Omega_a^{\cdot} M$ then. Further, one can see that $\Phi_a(c(x)\alpha) = x \diamond \Phi_a(\alpha)$, for $x \in \mathrm{C}(g)_a$ and $\alpha \in \Omega_a^{\cdot} M$.

One should be cautioned that the above isomorphism Φ_a is not an algebra morphism. But it can be used to view the Clifford algebra $\mathrm{C}(g)_a$ as the differential algebra $\Omega_a^{\cdot} M$ endowed with a different multiplication. This is in fact the approach that we will take. There are several advantages; apart from being more explicit, so is one advantage that the set of tensors $\Omega_a^{\cdot} M$ is leaved unchanged by a metric variation, so one only has to consider the variation of the Clifford multiplication. In addition, even if one is interested in working only with the Clifford algebra $\mathrm{C}(g)_a$, experience shows that one has to work with products of elements of $\mathrm{C}(g)_a$ in a way corresponding to operations of $\Omega_a^{\cdot} M$ via the isomorphism Φ_a above.

So we will define the Clifford algebra $\mathrm{C}(g)_a$ to be the vectorspace $\Omega_a^{\cdot} M$ endowed with a different multiplication, the Clifford multiplication, which will be denoted by $\diamond$. But before this can be done, we will have to introduce some notation.

Just as above, for $x \in M_a^*$ and $\alpha \in \Omega_a^{\cdot} M$, $e(x).\alpha := x \wedge \alpha$ denotes exterior multiplication, and the metric dual $i(x).\alpha$ is called the interior multiplication, and is defined by the formula $g(e(x).\beta, \alpha) :=: g(\beta, i(x).\alpha)$, where $\beta \in \Omega_a^{\cdot} M$ is another form. For homogeneous α one has $\deg e(x).\alpha = \deg \alpha + 1$ and $\deg i(x).\alpha = \deg \alpha - 1$.

It is possible to give an explicit formula for the interior multiplication $i(x)$, and it is

$$i(x).\,(\alpha_1 \wedge \ldots \wedge \alpha_p) = \sum_{k=1}^{p} (-1)^{k-1} \alpha_1 \wedge \ldots \wedge \widehat{\alpha_k} \wedge \ldots \wedge \alpha_p \tag{4.19}$$

where $\widehat{\alpha_k}$ means that α_k is to be omitted from the product in which it stands, and the $\alpha_k \in \Omega^1_a M$ are of degree one. There is nothing that prevents one from having forms $x \in \Omega^{\cdot}_a M$ of higher degree in the definition of $e(x).\alpha := x \wedge \alpha$ and $i(x).\alpha$; in the case of $i(x).\alpha$ the corresponding explicit formula is $i(x_1 \wedge \ldots \wedge x_q).\alpha = i(x_q) \ldots i(x_1).\alpha$, for $x_k \in \Omega^1_a M$ of degree one, which is to be computed recursively then. (The expansion of this formula is written down in Bourbaki *Algebra* [4], ch III, §11.9, equation (68), p 604.)

Note that for $x \in \Omega^1_a M$, one has $e(x)^2 = 0$, and because of this, one also has $i(x)^2 = 0$. In addition, one can show that for any $x, y \in \Omega^1_a M$, $\{e(x), i(y)\} = g(x, y)$, where $\{A, B\} := AB + BA$ is the anticommutator, which is a useful relation.

Next, for $x \in M^*_a$, define, as above, the Clifford multiplication $c(x) := e(x) + i(x)$; clearly this is an $\mathbb{R}$ linear map, so in order to see that it extends to all $x \in \mathrm{C}(g)_a$, the thing to verify is that $c(x)^2 = q(x)$. But $c(x)^2 = e(x)^2 + i(x)^2 + \{e(x), i(x)\} = g(x, x) = q(x)$, so this is clear.

This Clifford multiplication is then extended to arbitrary elements $x \in \mathrm{C}(g)_a$ by the formula

$$
\begin{aligned}
c\,(x_1 \circ \ldots \circ x_q).\alpha &= c(x_1) \ldots c(x_q).\alpha \\
&= (e(x_1) + i(x_1)) \ldots (e(x_q) + i(x_q)).\alpha
\end{aligned}
\tag{4.20}
$$

where the $x_k \in \mathrm{C}(g)^1_a = \Omega^1_a M = M^*_a$.

In order to treat $\mathrm{C}(g)_a$ as the vectorspace $\Omega^{\cdot}_a M$ endowed with a different multiplication $\circ$, we need to know how to convert elements which are the sums of elements of the form $x_1 \circ \ldots \circ x_p$ to sums of elements of the form $\alpha_1 \wedge \ldots \wedge \alpha_p$, and vice versa, where the $x_k, \alpha_k \in \mathrm{C}(g)^1_a = \Omega^1_a M = M^*_a$. The first part is easy, we apply the above formula, setting $\alpha = 1$, which gives

$$
\begin{aligned}
c\,(x_1 \circ \ldots \circ x_q).1 &= c(x_1) \ldots c(x_q).1 \\
&= (e(x_1) + i(x_1)) \ldots (e(x_q) + i(x_q)).1
\end{aligned}
\tag{4.21}
$$

Here, the left hand side may be identified with the element $x_1 \circ \ldots \circ x_q \in \mathrm{C}(g)_a$, whereas the right hand side is clearly an element in $\Omega^{\cdot}_a M$. In fact, one can show that this formula gives the inverse $\Phi_a^{-1} \colon \mathrm{C}(g)_a \to \Omega^{\cdot}_a M$ to the map Φ_a discussed above.

In our situation where $\mathrm{C}(g)_a$ and $\Omega^{\cdot}_a M$ are identified as underlying vectorspaces, the formula simply reads

$$
\begin{aligned}
x_1 \circ \ldots \circ x_q &= c(x_1) \ldots c(x_q).1 \\
&= (e(x_1) + i(x_1)) \ldots (e(x_q) + i(x_q)).1
\end{aligned}
\tag{4.22}
$$

For example, $x \circ y = (e(x)+i(x))(e(y)+i(y)).1 = (e(x)+i(x)).y = x \wedge y + g(x, y)$. Note that, since $x \circ y = \frac{1}{2}[x, y] + \frac{1}{2}\{x, y\} = \frac{1}{2}[x, y] + g(x, y)$, where $[A, B] := A \circ B - B \circ A$ is the commutator and $\{A, B\} := A \circ B + B \circ A$ is the anticommutator in the Clifford algebra $C(g)_a$, this identifies $x \wedge y = \frac{1}{2}[x, y]$ as elements of this Clifford algebra; below, in formula (4.30), we shall return to the generalization of this formula. And for three elements $x, y, z \in C(g)^1_a = M^*_a$, formula (4.22) reads

$$x \circ y \circ z = (e(x) + i(x)).(y \wedge z + g(y, z)) \\ = x \wedge y \wedge z + g(y, z)x - g(x, z)y + g(x, y)z \tag{4.23}$$

We also want to be able to convert tensors which are sums with terms of the form $\alpha_1 \wedge \ldots \wedge \alpha_p$ to elements which are sums of elements of the form $x_1 \circ \ldots \circ x_p$, with $x_k, \alpha_k \in C(g)^1_a = \Omega^1_a M = M^*_a$, as above. This can be done just as easily, but it seems appropriate to state the principle hidden in this: The differential algebra $\Omega^{\cdot}_a M$ can be considered as a Clifford algebra with 0 (zero) metric, and the isomorphism Φ_a is then a homomorphism between Clifford algebras of relative metric $g-0 = g$. It is in general possible, for any two metrics g_1 and g_2, to construct an isomorphism $\Phi(h) : C(g_1) \to C(g_2)$ of relative metric $h := g_2 - g_1$, and one can show that for compositions $\Phi(h_1) \circ \Phi(h_2) = \Phi(h_1 + h_2)$, where h_1 and h_2 are two relative metrics then. Moreover, the isomorphism of the relative metric zero $\Phi(0)$ is equal to the identity of any Clifford algebra $C(g)$, so it follows that $\Phi(h)^{-1} = \Phi(-h)$. In other words, we will expect $\Omega^{\cdot}_a M$ to behave as a Clifford algebra of relative metric $-g$, relative to the Clifford $C(g)_a$

First, introduce the following Clifford algebra interior multiplication

$$i(x).\,(\alpha_1 \circ \ldots \circ \alpha_p) = \sum_{k=1}^{p} (-1)^{k-1} \alpha_1 \circ \ldots \circ \widehat{\alpha_k} \circ \ldots \circ \alpha_p \tag{4.24}$$

where $x \in M^*_a$ and the $\alpha_k \in C(g)^1_a = \Omega^1_a M = M^*_a$. In order to see that this indeed defines a map $i(x)\colon C(g)_a \to C(g)_a$, one first views it as a map $i(x)\colon T(M^*_a) \to C(g)_a$, defined by the same formula, but with $\circ$ replaced by $\otimes$ on the left hand side. Then, in order to show that this map factors over $C(g)_a$, one needs only to show that $i(x)$ maps the ideal in $T(M^*_a)$ generated by elements of the form $y \otimes y - q(y)$, for $y \in M^*_a$, into zero in $C(g)_a$. But $i(x).(y \otimes y - q(y)) = g(x, y)y - g(x, y)y - 0 = 0$, so this is clear.

The formula given for $i(x)$ operating on $C(g)_a$ is the same formula as the explicit one above given for the $i(x)$ operating on $\Omega^{\cdot}_a M$, with the difference that it is operating only on Clifford products. Actually, the two $i(x)$ are the same, in the sense that one can freely express any element of the vectorspace $\Omega^{\cdot}_a M = C(g)_a$ as a sum of terms being either Clifford-products $\alpha_1 \circ \ldots \circ \alpha_p$ or exterior products $\beta_1 \wedge \ldots \wedge \beta_q$, with $\alpha_k, \beta_k \in C(g)^1_a = \Omega^1_a M = M^*_a$; after that one uses on each term, the version of $i(x)$ that is appropriate. This follows, using the above isomorphism $\Phi_a\colon \Omega^{\cdot}_a M \xrightarrow{\cong} C(g)_a$, because one can show that $\Phi_a(i(x).\alpha) = i(x).\Phi_a(\alpha)$, for any $\alpha \in \Omega^{\cdot}_a M$. In our picture, $C(g)_a$ and $\Omega^{\cdot}_a M$ are identified, so α and $\Phi_a(\alpha)$ will actually represent the same element, α expressed as sums of exterior products, and $\Phi_a(\alpha)$ expressed as sums of Clifford products. In order to show that $\Phi_a(i(x).\alpha) = i(x).\Phi_a(\alpha)$, one conveniently relies on an abstract machinery; basically this happens because the $i(x)$ and Φ_a are induced from the corresponding notions on $\mathrm{T}(M_a^*)$, and there is only one way to define it over $\mathrm{T}(M_a^*)$, see Bourbaki *Algebra* [4], chIX, §9.2, lemma 2.

Returning to the question on how to convert tensors which are sums with terms of the form $\alpha_1 \wedge \ldots \wedge \alpha_p$ to elements which are sums of elements of the form $x_1 \square \ldots \square x_p$, with $x_k, \alpha_k \in \mathrm{C}(g)_a^1 = \Omega_a^1 M = M_a^*$ then. If we would expect $\Omega_a^{\cdot} M$ to behave as a Clifford algebra with relative metric $-g$, relative to $\mathrm{C}(g)_a$, then inverting the above formula for turning Clifford multiplications into exterior products, we would expect it to be

(4.25)
$$\begin{aligned} e\left(x_1 \wedge \ldots \wedge x_q\right).\alpha &= e(x_1) \ldots e(x_q).\alpha \\ &= (c(x_1) - i(x_1)) \ldots (c(x_q) - i(x_q)).\alpha \end{aligned}$$

which is also the correct formula. Setting, as before, $\alpha = 1$ then gives

$$\begin{aligned} x_1 \wedge \ldots \wedge x_q &= e(x_1) \ldots e(x_q).1 \\ &= (c(x_1) - i(x_1)) \ldots (c(x_q) - i(x_q)).1 \end{aligned} \tag{4.26}$$

For instance $x \wedge y = (c(x) - i(x))(c(y) - i(y)).1 = (c(x) - i(x)).y = x \square y - g(x, y)$, or $x \wedge y = x \square y - g(x, y)$, which is the same result as we got when converting $x \square y$ into $x \wedge y + g(x, y)$ above. And for three elements $x, y, z \in \mathrm{C}(g)_a^1 = M_a^*$, we get

$$\begin{aligned} x \wedge y \wedge z &= (c(x) - i(x)).(y \square z - g(y, z)) \\ &= x \square y \square z - g(y, z)x + g(x, z)y - g(x, y)z \end{aligned} \tag{4.27}$$

also this the same result as above.

This completes the description of $\mathrm{C}(g)_a$ as the vectorspace $\Omega_a^{\cdot} M$ endowed with a different multiplication, the Clifford multiplication, because we can freely and at need use the above formulas to convert between sums of Clifford product terms and sums of exterior product terms.

However there is another interesting formula, worth to be mentioned:

For a product $\cdot$ of some kind in an algebra $\mathcal{A}$, introduce the higher commutators

$$[x_1, \ldots, x_p] := \sum_{\sigma} \operatorname{sgn} \sigma \;\; x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(p)} \tag{4.28}$$

where the $x_k \in \mathcal{A}$, and σ is a permutation of $(1, \ldots, p)$. When clarity so demands, the kind of product $\cdot$ involved will be denoted as a subscript, for example $[\ldots]_\wedge$ for the exterior product and $[\ldots]_\square$ for the Clifford multiplication.

Then clearly, for $\alpha_k \in \Omega_a^1 M$,

$$\alpha_1 \wedge \ldots \wedge \alpha_p = \frac{1}{p!} \left[\alpha_1 \, \ldots \, \alpha_p\right]_\wedge \tag{4.29}$$

But it is interesting to note that a similar formula[36] holds for the higher Clifford multiplication commutator:

$$\alpha_1 \wedge \ldots \wedge \alpha_p = \frac{1}{p!}\,[\alpha_1 \ \ldots \ \alpha_p]_{\circ} = \frac{1}{p!}\sum_{\sigma} \operatorname{sgn}\sigma \ \ \alpha_{\sigma(1)} \circ \ldots \circ \alpha_{\sigma(p)} \tag{4.30}$$

where the $\alpha_k \in \mathrm{C}(g)^1_a = \Omega^1_a M = M^*_a$, and σ is a permutation of $(1, \ldots, p)$.

One way to prove this formula, is by first taking an orthonormal basis $\omega^\mu \in M^*_a$, and setting $\alpha_k = \omega^{\mu_k}$, for $\mu_1 < \ldots < \mu_p$ (possible only if $p \leq \dim M$), in which case the Clifford multiplication reduces to exterior multiplication, and the formula is obviously true. The general case then follows by observing that both the left hand side and right hand side are multilinear over $\mathbb{R}$ and antisymmetrical.

One can also give a direct, computational proof of the above formula. I include a proof, because it contains some interesting computations:

On the Clifford algebra $\mathrm{C}(g)_a$, for $x, \alpha_k \in M^*$, the objective is to show that

$$\begin{aligned} \frac{1}{(p+1)!}[x, \alpha_1, \ldots, \alpha_p]_{\circ} &= x \circ \frac{1}{p!}\,[\alpha_1, \ldots, \alpha_p]_{\circ} - i(x).\frac{1}{p!}\,[\alpha_1, \ldots, \alpha_p]_{\circ} \\ &= (c(x) - i(x)).\frac{1}{p!}\,[\alpha_1, \ldots, \alpha_p]_{\circ} \end{aligned} \tag{4.31}$$

This formula can then be used to inductively show that $\alpha_1 \wedge \ldots \wedge \alpha_p = \frac{1}{p!}\,[\alpha_1, \ldots, \alpha_p]_{\circ}$; because of the relation $e(x) = c(x) - i(x)$, if this true for p elements, then formula (4.31) shows it is also true for $p+1$ elements.

Introduce some abbreviational notation:

$$\begin{cases} \alpha := \dfrac{1}{p!}\,[\alpha_1, \ldots, \alpha_p]_{\circ} \\ \alpha_{\hat{k}} := \dfrac{1}{(p-1)!}[\alpha_1, \ldots, \widehat{\alpha_k}, \ldots, \alpha_p]_{\circ} \end{cases} \tag{4.32}$$

Then, in order to show formula (4.31) above, first observe that

$$[\alpha_1, \ldots, \alpha_p]_{\circ} = \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \circ [\alpha_1, \ldots, \widehat{\alpha_k}, \ldots, \alpha_p]_{\circ} \tag{4.33}$$

which follows by collecting terms starting with α_k in the expansion $[\alpha_1, \ldots, \alpha_p] = \sum_\sigma \operatorname{sgn}\sigma \ \alpha_{\sigma(1)} \circ \ldots \circ \alpha_{\sigma(p)}$, and is thus true for any multiplication, not only for Clifford multiplication $\circ$.

[36] See Bourbaki, *Algebra* [4], ch IX, §9, exercise 3c.

Using the shorthand notation (4.32) above, formula (4.33) can also be written

$$p\alpha = \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \,\Box\, \alpha_{\hat{k}} \tag{4.34}$$

From formula (4.34), we then see that

$$(p-1) i(x).\alpha = - \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \,\Box\, i(x).\alpha_{\hat{k}} \tag{4.35}$$

This follows, because

(4.36)

$$\begin{aligned} p\, i(x).\alpha &= i(x).(p\alpha) \\ &= \sum_{k=1}^{p} (-1)^{k-1} i(x).(\alpha_k \,\Box\, \alpha_{\hat{k}}) \\ &= \sum_{k=1}^{p} (-1)^{k-1} (i(x).\alpha_k) \alpha_{\hat{k}} - \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \,\Box\, i(x).\alpha_{\hat{k}} \\ &= i(x).\alpha - \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \,\Box\, i(x).\alpha_{\hat{k}} \end{aligned}$$

In the last equality here, we used

$$i(x).\alpha = \sum_{k=1}^{p} (-1)^{k-1} (i(x).\alpha_k) \alpha_{\hat{k}} \tag{4.37}$$

This follows, because first

$$i(x).\, \alpha_1 \,\Box\, \dots \,\Box\, \alpha_p = \sum_{k=1}^{p} (-1)^{k-1} (i(x).\alpha_k) \alpha_1 \,\Box\, \dots \,\Box\, \widehat{\alpha_k} \,\Box\, \dots \,\Box\, \alpha_p \tag{4.38}$$

and from this equation, by taking the signed summation $\sum_\sigma$ over permutations σ of $(1, \dots, p)$, one then gets

$$i(x).\, [\alpha_1, \dots, \alpha_p]_{\Box} = \sum_{k=1}^{p} (-1)^{k-1} p (i(x).\alpha_k) [\alpha_1, \dots, \widehat{\alpha_k}, \dots, \alpha_p]_{\Box} \tag{4.39}$$

Dividing so both sides with $p!$, gives formula (4.37).

Returning so to the proof of formula (4.31) above, that is

$$\frac{1}{(p+1)!} [x, \alpha_1, \dots, \alpha_p]_{\Box} = (c(x) - i(x)).\alpha \tag{4.40}$$

the proof is done by induction, assuming it to be true for $p-1$.

One then has

$$\frac{1}{(p+1)!}[x, \alpha_1, \dots, \alpha_p]_{\scriptstyle\square} = \frac{1}{p+1}(x \mathbin{\scriptstyle\square} \alpha - \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \mathbin{\scriptstyle\square} \frac{1}{p!}[x, \alpha_1, \dots, \widehat{\alpha_k}, \dots, \alpha_p])$$

$$\text{(by induction)} \quad = \frac{1}{(p+1)}(x \mathbin{\scriptstyle\square} \alpha - \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \mathbin{\scriptstyle\square} (x \mathbin{\scriptstyle\square} \alpha_{\hat{k}} - i(x).\alpha_{\hat{k}}))$$

and using $\alpha_k \mathbin{\scriptstyle\square} x = -x \mathbin{\scriptstyle\square} \alpha_k + 2g(x, \alpha_k)$, we continue

$$\begin{aligned}
&= \frac{1}{p+1}(x \mathbin{\scriptstyle\square} \alpha - \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \mathbin{\scriptstyle\square} x \mathbin{\scriptstyle\square} \alpha_{\hat{k}} + \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \mathbin{\scriptstyle\square} i(x).\alpha_{\hat{k}})) \\
&= \frac{1}{p+1}(x \mathbin{\scriptstyle\square} \alpha + x \mathbin{\scriptstyle\square} \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \mathbin{\scriptstyle\square} \alpha_{\hat{k}} - 2 \sum_{k=1}^{p} (-1)^{k-1} g(x, \alpha_k) \alpha_{\hat{k}} \\
&\qquad\qquad + \sum_{k=1}^{p} (-1)^{k-1} \alpha_k \mathbin{\scriptstyle\square} i(x).\alpha_{\hat{k}}) \\
&= \frac{1}{p+1}(x \mathbin{\scriptstyle\square} \alpha + p\, x \mathbin{\scriptstyle\square} \alpha - 2i(x).\alpha + (p-1) i(x).\alpha) \\
&= x \mathbin{\scriptstyle\square} \alpha - i(x).\alpha
\end{aligned}$$

as we wanted to see.

For two elements $x, y \in M_a^*$, formula (4.30) above just says that $x \wedge y = \frac{1}{2}(x \mathbin{\scriptstyle\square} y - y \mathbin{\scriptstyle\square} x)$, which is easy enough to see, since $x \wedge y = x \mathbin{\scriptstyle\square} y - g(x, y)$. It is considerably more difficult to directly work out the corresponding formula for three elements; in fact, one ends up working through the computations in the proof above.

5 Tensor Formulas

Above, in section 2, some coordinate dependant spin operators was introduced. Before the corresponding coordinate free version can be presented, which will be done in section 6, it will be necessary to review some of the formulas to be used.

Let A and B be tensors of type (p, q), so that

$$A = \sum_{\substack{\mu_1 \cdots \mu_p \\ \nu_1 \cdots \nu_p}} A^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} \, \tfrac{\partial}{\partial x^{\mu_1}} \otimes \ldots \otimes \tfrac{\partial}{\partial x^{\mu_p}} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_q} \tag{5.1}$$

or, for short, $A = A^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$, and similarly for the tensor B.

Then the metric $g(A, B)$ is defined by the formula

$$g(A, B) := A^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} g_{\mu_1 \hat{\mu}_1} \ldots g_{\mu_p \hat{\mu}_p} g^{\nu_1 \hat{\nu}_1} \ldots g^{\nu_p \hat{\nu}_p} B^{\hat{\mu}_1 \cdots \hat{\mu}_p}{}_{\hat{\nu}_1 \cdots \hat{\nu}_q} \tag{5.2}$$

which, with tensor shifting conventions, can be written as

$$g(A, B) = A^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} B_{\mu_1 \cdots \mu_p}{}^{\nu_1 \cdots \nu_q} = A_{\mu_1 \cdots \mu_p}{}^{\nu_1 \cdots \nu_q} B^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$$

and so on.

If A and B are of different type, that is, say A is of type (p, q) and B is of type (p', q') with $(p, q) \neq (p', q')$ as number pairs, then, by definition, $g(A, B) := 0$. Of course, it is not necessary to contract all indices simultaneously, but this is being handled with other notation, depending on the situation.

On the other hand, if $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_p \in \Omega^1 M$ are one-forms (degree one cotensors), then the corresponding metric of their exterior products is defined by the formula

$$g\left(\alpha_1 \wedge \ldots \wedge \alpha_p \, , \, \beta_1 \wedge \ldots \wedge \beta_p\right) := \det(g(\alpha_j, \beta_k))_{j,k=1,\ldots,p} \tag{5.3}$$

This is then being extended to all differential forms by means of linear extension, using the additional rule that for differential forms of different degree, $\alpha \in \Omega^p M$ and $\beta \in \Omega^q M$ with $p \neq q$, their metric product is zero, $g(\alpha, \beta) := 0$.

Suppose now that the tensors A and B above are antisymmetric cotensors of degree p, that is, tensors of type $(0, p)$. Then, as we saw in the previous section[37] above, they can be identified with elements of $\Omega^p M$ according to the following formula

$$\begin{aligned} A &= A_{\mu_1 \ldots \mu_p} \, dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_p} \\ &= \sum_{\mu_1 < \ldots < \mu_p} A_{\mu_1 \ldots \mu_p} \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \\ &= \frac{1}{p!} \sum_{\mu_1, \ldots, \mu_p} A_{\mu_1 \ldots \mu_p} \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \end{aligned} \tag{5.4}$$

[37]See equation (4.12).

and there is a similar formula for the tensor B then. If the metric $g(A,B)$ is computed when A and B are considered as tensors, we get the following formula (with an index $\otimes$ used to indicate that the metric is computed as tensors only)

$$g(A,B)_\otimes = \sum_{\mu_1,\dots,\mu_p} A_{\mu_1\cdots\mu_p} g^{\mu_1\hat\mu_1}\dots g^{\mu_p\hat\mu_p} B_{\hat\mu_1\cdots\hat\mu_p} \tag{5.5}$$

On the hand, considering A and B as differential forms (as elements of $\Omega^p M$) gives the formula (with an index Ω used to indicate that the metric is computed as differential forms)

$$\begin{aligned} g(A,B)_\Omega &= \sum_{\substack{\mu_1<\cdots<\mu_p \\ \hat\mu_1<\cdots<\hat\mu_p}} A_{\mu_1\cdots\mu_p} g^{\mu_1\hat\mu_1}\dots g^{\mu_p\hat\mu_p} B_{\hat\mu_1\cdots\hat\mu_p} \\ &= \frac{1}{(p!)^2}\sum_{\substack{\mu_1\cdots\mu_p \\ \hat\mu_1\cdots\hat\mu_p}} A_{\mu_1\cdots\mu_p} g^{\mu_1\hat\mu_1}\dots g^{\mu_p\hat\mu_p} B_{\hat\mu_1\cdots\hat\mu_p} \\ &= \frac{1}{(p!)^2} g(A,B)_\otimes \end{aligned} \tag{5.6}$$

In other words, since the outcome of the metric computations of the antisymmetrical tensors A and B differ, depending on whether A and B are considered as tensors only, or as differential forms, one has to be somewhat careful as with respect to which situation you are in. One might be tempted changing the definition of the metric $g(A,B)_\Omega$ in order to avoid this problem, but then one will have to insert the corresponding factors in computations within $\Omega^\cdot M$, like when using the interior product $i(x)$, the Hodge star operator $*$, and in some other situations.

The convention here gives that when computing the metrics of the basis element on $\Omega^\cdot M$ built up by exterior products from a local orthonormal basis dx^μ, one has

$$\begin{cases} g\left(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}\ ,\, dx^{\mu_{\sigma(1)}}\wedge\ldots\wedge dx^{\mu_{\sigma(p)}}\right) = \operatorname{sgn}\sigma\, g^{\mu_1\mu_1}\dots g^{\mu_p\mu_p} \\ g\left(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}\ ,\, dx^{\nu_1}\wedge\ldots\wedge dx^{\nu_p}\right) = 0, \quad \{\mu_1,\dots,\mu_p\}\neq\{\nu_1,\dots,\nu_p\} \end{cases}$$

where σ is a permutation of $(1,\dots,p)$, where the μ_k are assumed to be distinct, and where the ν_k are assumed to be distinct. This clearly avoids any unnecessary factors.

In physics, the common thing is to work with tensor components only, which leads to the use of the metric $g(A,B)_\otimes$ only, and special attention has to be given to the insertion of suitable dualizing constants.

5.1 Tensor Operations

Let $\mathcal{A}$ be a graded algebra, with $\cdot$ being its multiplication; in practise $\mathcal{A}$ will typically be one (or a suitable derivation) of the following algebras: The bigraded tensor algebra $\mathrm{T}^{\cdot,\cdot}M$ with multiplication $\otimes$ and bidegree being the tensor type (p,q) and total degree defined to be $p+q$, the singly graded differential algebra $\Omega^{\cdot}M$ with multiplication $\wedge$, the singly graded symmetric algebra $\mathrm{S}^{\cdot}M := \mathrm{S}(\mathrm{T}^{*}M)$ with multiplication $\odot$, or the $\mathbb{Z}_2$ graded Clifford algebra $\mathrm{C}(g)$ with multiplication $\circ$ and the grading being $\{\mathrm{even}, \mathrm{odd}\}$.

A linear map $B\colon \mathcal{A} \to \mathcal{A}$ is said to be (homogeneous) of degree b if it happens that it sends (homogeneous) elements of degree a to (homogeneous) elements of degree $a+b$.

A linear map $B\colon \mathcal{A} \to \mathcal{A}$ is called a derivation if it satisfies the relation $B(x\cdot y) = (Bx)\cdot y + x\cdot By$. Alternatively, a map A satisfying such a relation will be said to act by Lie algebra action on the product $x\cdot y$, since this kind of action is the one to use for Lie algebras acting on the tensor products of their representations.

This is to be distinguished from a linear map $C\colon \mathcal{A} \to \mathcal{A}$ satisfying the relation $C(x\cdot y) = (Cx)\cdot(Cy)$, which will be called a diagonal, or group action. It is easy to see that if A acts by Lie algebra action, then $C := \exp A := \sum_{k=0}^{\infty} \frac{1}{k!}A^k$, then $C := \exp A$ acts by group action. Vice versa, given a group action defined via such an exponential $\exp A$, the Lie algebra action is retrived by the formula $A = \frac{d}{ds}\exp(sA)\big|_{s=0}$; the set of elements of the form $\{\exp(sA) \mid s \in \mathbb{R}\}$ is called a one-parameter group.

But it is also frequent that a linear map $B\colon \mathcal{A} \to \mathcal{A}$ satisfies the relation $B(x\cdot y) = (Bx)\cdot y + (-1)^{\deg x} x\cdot By$ (the way to remember this kind of signed relations is that the exponent of the appearing (-1) of a particular term consists of the products of the the degrees of the elements or functions that pass over each other in that term, in this case $\deg x$ and $\deg B \overset{(2)}{\equiv} 1$); it is then called an antiderivation.

The interior multiplication $i(x)$ is an antiderivation; it can be constructed as follows (see Bourbaki *Algebra* [4], ch IX, §9.2 for details): Take a $\mathcal{C}^\infty M$ linear function $f\colon T^{*}M \to \mathbb{R}$, which is an element in the dual of $\mathrm{T}^{*}M$; of course, f can be identified with a vectorfield X by requiring the formula $f(\alpha) =: \alpha(X)$ to hold for all $\alpha \in \mathrm{T}^{*}M$, which we will write $f_X := f$. Then define

$$
\begin{aligned}
(5.7) \qquad i(f).\,(\alpha_1 \otimes \ldots \otimes \alpha_p) &:= \sum_{k=1}^{p} (-1)^k f(\alpha_k)\alpha_1 \otimes \ldots \otimes \widehat{\alpha_k} \otimes \ldots \otimes \alpha_p \\
&= \sum_{k=1}^{p} (-1)^k \alpha_k(X)\alpha_1 \otimes \ldots \otimes \widehat{\alpha_k} \otimes \ldots \otimes \alpha_p
\end{aligned}
$$

where the $\alpha_k \in \mathrm{T}^{*}M$, and where the second equality uses $f := f_X$. There is also the alternative notation $i_f := i(f)$, and also $i(X) := i(f_X)$.

For $x \in \mathrm{T}^{*}M$, let $t(x)$ be the tensor product, so that for an arbitrary tensor $y \in \mathrm{T}^{\cdot,\cdot}M$ one has $t(x).y := x\otimes y$. Then the tensor product $t(x)$ and the interior product satisfies the following relations

$$\begin{cases} \{t(x), i(f)\} = f(x) \\ i(f).1 = 0 \end{cases} \tag{5.8}$$

where $\{A, B\} := AB + BA$ is the notation for the anticommutator, and $f(x)$ is really short for $f(x).1$, where 1 is the identity map. Alternatively, one can take these two equations in formula (5.8) as a recursive definition of the interior multiplication $i(f)$. In addition one has the relations

$$\begin{cases} \{i(f_1), i(f_2)\} = 0 \\ i(f)^2 = 0 \end{cases} \tag{5.9}$$

for any $\mathcal{C}^\infty M$ linear maps $f, f_1, f_2 \colon T^*M \to \mathbb{R}$.

One can then show that this interior multiplication $i(f)$ factors to any Clifford algebra $\mathrm{C}(g)$, and in particular, setting $g = 0$, it factors to the differential algebra $\Omega^{\cdot} M$, the formulas being the same as above in equation (5.7), except that one in the respective cases replaces the tensor product $\otimes$ with the Clifford product $\scriptstyle\square$ or the exterior product $\wedge$.

Now, if $x \in \mathrm{T}^*M$, we can use the metric to associate a dual vectorfield X by requiring that, for all vectorfields Y, the formula $x(Y) :=: g(X, Y)$ to hold; alternatively, we can define a linear map $f \colon T^*M \to \mathbb{R}$ by requiring $f(y) :=: g(x, y)$ to hold for all cotensors $y \in \mathrm{T}^*M$, and then, of course $f = f_X$. Applying this, results in the more familiar looking formulas

$$i(x).\,(\alpha_1 \otimes \ldots \otimes \alpha_p) = \sum_{k=1}^{p} (-1)^k g(x, \alpha_k) \alpha_1 \otimes \ldots \otimes \widehat{\alpha_k} \otimes \ldots \otimes \alpha_p \tag{5.10}$$

$$i(x).\,(\alpha_1 \wedge \ldots \wedge \alpha_p) = \sum_{k=1}^{p} (-1)^k g(x, \alpha_k) \alpha_1 \wedge \ldots \wedge \widehat{\alpha_k} \wedge \ldots \wedge \alpha_p \tag{5.11}$$

$$i(x).\,(\alpha_1 \,\scriptstyle\square\, \ldots \,\scriptstyle\square\, \alpha_p) = \sum_{k=1}^{p} (-1)^k g(x, \alpha_k) \alpha_1 \,\scriptstyle\square\, \ldots \,\scriptstyle\square\, \widehat{\alpha_k} \,\scriptstyle\square\, \ldots \,\scriptstyle\square\, \alpha_p \tag{5.12}$$

The corresponding anticommutator relations then become, for $x, y \in \mathrm{T}^*M$,

$$\{t(x), i(y)\} = g(x, y), \quad \{e(x), i(y)\} = g(x, y), \quad \{c(x), i(y)\} = g(x, y) \tag{5.13}$$

where last equation follows from that $\{i(x), i(y)\} = 0$.

As has been remarked before, the interior multiplication $i(x)$ acting on the differential algebra $\Omega^{\cdot} M$ is the metric dual of the exterior multiplication $e(x)$, that is, $g(e(x).\alpha, \beta) = g(\alpha, i(x).\beta)$ for all $\alpha, \beta \in \Omega^{\cdot} M$ (and if α and β are homogeneous, in order for the both sides of this equation to be non-zero, one must have $\deg \beta = 1 + \deg \alpha$). Alternatively, one can take this equation as the defining equation for the action of $i(x)$ on $\Omega^{\cdot} M$, in which case one also automatically gets a definition of $i\,(x_1 \wedge \ldots \wedge x_p) = i(x_p)\ldots i(x_1)$ for $x_k \in \mathrm{T}^*M$, as one can see (this was remarked in greater detail above).

This was concerning multiplication $e(x).\alpha := x \wedge \alpha$ from the left; one can also consider multiplication from the right $\alpha.\overleftarrow{e}(x) := \alpha \wedge x$. This will yield a corresponding $\overleftarrow{\imath}(x)$, defined by the equation $g(\alpha.\overleftarrow{e}(x), \beta) =: g(\alpha, \beta.\overleftarrow{\imath}(x))$. Since, if $x \in \Omega^p M$ and $\alpha \in \Omega^q M$, one has $e(x).\alpha := x \wedge \alpha = (-1)^{pq}\alpha \wedge x = (-1)^{pq}\alpha.\overleftarrow{e}(x)$, it follows that

$$\begin{aligned} g(\alpha, \beta.\overleftarrow{\imath}(x)) = g(\alpha.\overleftarrow{e}(x), \beta) &= g(\alpha \wedge x, \beta) \\ &= (-1)^{pq} g(x \wedge \alpha, \beta) = (-1)^{pq} g(e(x).\alpha, \beta) \\ &= (-1)^{pq} g(\alpha, i(x).\beta) \end{aligned} \tag{5.14}$$

and applying this to the only relevant case $\deg\beta = \deg x + \deg\alpha = p + q$, we get

$$\begin{aligned} \beta.\overleftarrow{\imath}(x) &= (-1)^{\deg x(\deg\beta - \deg x)} i(x).\beta \\ &= (-1)^{\deg x(\deg\beta - 1)} i(x).\beta \end{aligned} \tag{5.15}$$

where we used the fact that $(\deg x)^2 \overset{(2)}{\equiv} \deg x$.

For instance, if $\deg x = 1$, this gives

$$\beta.\overleftarrow{\imath}(x) = (-1)^{\deg\beta - 1} i(x).\beta \tag{5.16}$$

One would think that because of such a formula (equation (5.15)), the interior multiplication from the right $\overleftarrow{\imath}(x)$ is a wholly unnecessary concept, but it shows up naturally in the context of Clifford multiplications from the right, and it is then convenient to use $\overleftarrow{\imath}(x)$ without having to worry about inserting the correct minus signs.

There is a formula $\overleftarrow{\imath}\,(x_1 \wedge \ldots \wedge x_p) = \overleftarrow{\imath}(x_p)\ldots\overleftarrow{\imath}(x_1)$ for $x_k \in \mathrm{T}^*M$, similar to the one for the left interior multiplication $i(x)$, giving a method to compute $\overleftarrow{\imath}(x)$ recursively for any $x \in \Omega^{\cdot} M$.

Instead of writing the right multiplications to the right, one can also write them to the left, that is $e_r(x).\alpha := \alpha.\overleftarrow{e}(x) := \alpha \wedge x$, and similarly $i_r(x).\alpha := \alpha.\overleftarrow{\imath}(x)$. Doing this, the only thing one has to watch out for is the reversal of order, relative $e(x)$ and $i(x)$, in the formulas

$$e_r(x \wedge y) = e_r(y)e_r(x), \qquad i_r(x \wedge y) = i_r(x)i_r(y) \tag{5.17}$$

for $x, y \in \Omega^{\cdot} M$.

One should note that the Clifford multiplication $c(x) = e(x) + i(x)$ is self-dual with respect to the metric g, that is,

$$g(c(x).\alpha, \beta) = g(\alpha, c(x).\beta). \tag{5.18}$$

Also note that, since $e(x)$ raises the degrees of the tensors it acts on, and $i(x)$ correspondingly lowers those degrees, $c(x)$ does in general not preserve the homogenity of the integer valued degrees, but it does preserve the homogenity of the $\{\mathrm{even}, \mathrm{odd}\} = \mathbb{Z}_2$ degrees.

One can also define a Clifford multiplication from the right $\overleftarrow{c}(x) = \overleftarrow{e}(x) + \overleftarrow{\imath}(x)$. In the context of the isomorphism $\Phi\colon \Omega^{\cdot} \xrightarrow{\cong} \mathrm{C}(g)$ defined in the preceding section, it means that $\Phi(\alpha) \circ x = \Phi(\alpha.\overleftarrow{c}(x))$ for $\alpha \in \Omega^{\cdot} M$ and $x \in \mathrm{C}(g)$ (details omitted).

So far, we have only discussed the antiderivation $i(x)$ operating on the tensor algebra $\mathrm{T}^{0,\cdot}M$, the differential algebra $\Omega^{\cdot} M$, and the Clifford algebra $\mathrm{C}(g)$. It is possible to define the corresponding derivation $\iota(x)$ by simply dropping the factor $(-1)^k$ in the defining sums, that is,

$$\iota(x).\,(\alpha_1 \otimes \ldots \otimes \alpha_p) := \sum_{k=1}^{p} g(x, \alpha_k)\alpha_1 \otimes \ldots \otimes \widehat{\alpha_k} \otimes \ldots \otimes \alpha_p \tag{5.19}$$

for $x, \alpha_k \in \mathrm{T}^*M$.

This derivation then factors to the symmetric algebra $\mathrm{S}^{\cdot} M$, with the defining equation, except that the tensor $\otimes$ is being replaced by the corresponding symmetric tensor $\odot$, that is

$$\iota(x).\,(\alpha_1 \odot \ldots \odot \alpha_p) := \sum_{k=1}^{p} g(x, \alpha_k)\alpha_1 \odot \ldots \odot \widehat{\alpha_k} \odot \ldots \odot \alpha_p \tag{5.20}$$

for $x, \alpha_k \in \mathrm{T}^*M$.

The anticommutator relations, valid for the antiderivation $i(x)$, are replaced by the corresponding commutator relations

$$[\iota(x), t(y)] = g(x, y), \qquad [\iota(x), s(y)] = g(x, y) \tag{5.21}$$

for $x, y \in \mathrm{T}^*M$.

The derivation $\iota(x)$ is also the dual of the symmetric multiplication $s(x)$ operator, defined by $s(x).\alpha := x \odot \alpha$, that is,

$$g(s(x).\alpha, \beta) = g(\alpha, \iota(x).\beta) \tag{5.22}$$

Since the symmetric tensor product is commutative, multiplication from the right agrees with multiplication from the left, that is $\overleftarrow{s}(x) = s(x)$, and therefore also $\overleftarrow{\iota}(x) = \iota(x)$.

A for the spin theory in this paper most important set of derivations, is the set of the elements of the Lie algebra $\mathrm{o}(g)$. The Lie algebra (or rather the corresponding Lie algebra bundle) can be defined as the set of $\mathcal{C}^\infty M$ linear maps $A\colon \mathrm{T}M \to \mathrm{T}M$ satisfying $g(A.X, Y) + g(X, A.Y) = 0$ for all vectorfields $X, Y \in \mathrm{T}M$, but we shall be less interested in the action of $\mathrm{o}(g)$ on the vectorfields of $\mathrm{T}M$ than its action on the cotensors $\mathrm{T}^{0,\cdot}M$. As remarked above, each such Lie algebra element A can be extended to a namesake derivation $A\colon \mathrm{T}^{\cdot,\cdot} \to \mathrm{T}^{\cdot,\cdot}$, which simply corresponds to the tensorproduct of Lie algebra modules; in physics this is called the principle of superposition.

The Lie algebra $\mathrm{o}(g)$ can be identified with the set of differential two-forms $\Omega^2 M$, as follows:

First, let $a \wedge b \in \Omega^2$, act on $\alpha \in \Omega^{\cdot} M$ by the formula

$$S(a \wedge b).\alpha := (e(a)i(b) - e(b)i(a)).\alpha \tag{5.23}$$

which, because both sides are antisymmtrical in the pair (a, b), does define an action of $\Omega^2 M$. This formula is being introduced because it is convenient to work with in explicit computations. One can easily see that this defines a derivation on $\Omega^{\cdot} M$; for $\alpha, \beta \in \Omega^{\cdot} M$ one has

$$\begin{aligned} S(a \wedge b).(\alpha \wedge \beta) &:= (e(a)i(b) - e(b)i(a)).(\alpha \wedge \beta) \\ &= e(a).((i(b)\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i(b).\beta)) \\ &\quad + e(b).((i(a)\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i(a).\beta)) \\ &= ((e(a)i(b) - e(b)i(a)).\alpha) \wedge \beta + \alpha \wedge (e(a)i(b) - e(b)i(a)).\beta \\ &= (S(a \wedge b).\alpha) \wedge \beta + \alpha \wedge S(a \wedge b).\beta \end{aligned} \tag{5.24}$$

Now, if $x \in \Omega^1 M = \mathrm{T}^* M$, we have $S(a \wedge b).x = ag(b, x) - bg(a, x)$, so some straightforward computations show that $g(S(a \wedge b).x, y) + g(x, S(a \wedge b).y) = 0$ for all $x, y \in \mathrm{T}^* M$, that is $S(a \wedge b) \in \mathrm{o}(g)$. So this gives a linear map $S\colon \Omega^2 M \to \mathrm{o}(g)$.

On the other hand, given an element $A \in \mathrm{o}(g)$, which by definition satisfies the relation $g(A.x, y) + g(x, A.y) = 0$, we can associate an antisymmetrical linear function $f \in \Omega^2(\mathrm{T}M) =: \Omega^2 M$ as follows: Let a tilde $\tilde{}$ denote metric dualization using the metric g, so that for a vectorfield $X \in \mathrm{T}M$, we get the cotensor $\tilde{X}$ which by definition sends the vectorfield Y to $g(X, Y)$, and for a cotensor $x \in \mathrm{T}^* M$, we get the vectorfield $\tilde{x}$ which by definition is the (unique) vectorfield X satisfying $x(Y) = g(X, Y)$ for all vectorfields Y.

Then define $f(X, Y) := g(\tilde{X}, A.\tilde{Y})$, which is an antisymmetrical function because of the Lie algebra relation that A satisfies. It is easy to see that this defines the inverse of S, that is $S^{-1}(A) = f$:

In order to show (with S^{-1} temporarily only being the name of the function that sends A to f) that $S^{-1}(S(a \wedge b)) = a \wedge b$, using the identifications between tensors made above, we first note that $a \wedge b$ represents the function that sends the pair of vectorfields (X, Y) to $a(X)b(Y) - b(X)a(Y)$. Now $S^{-1}(S(a \wedge b))$ is the function that sends the pair of vectorfields (X, Y) to

$$\begin{aligned} g(\tilde{X}, S(a \wedge b).\tilde{Y}) &= g(\tilde{X}, ag(b, \tilde{Y}) - bg(a, \tilde{Y})) \\ &= g(\tilde{X}, a)g(b, \tilde{Y}) - g(b, \tilde{X})g(a, \tilde{Y}) \\ &= a(X)b(Y) - b(X)a(Y) \end{aligned} \tag{5.25}$$

And in order to show that $S(S^{-1}(A)) = A$, first note that an alternative description of S is $S(f).x = f(\cdot, \tilde{x}) : \mathrm{T}M \to \mathcal{C}^\infty M$; this follows by comparing the defining formula $S(a \wedge b).x := ag(b,x) - bg(a,x)$ with the formula $(a \wedge b)(X,Y) = a(X)b(Y) - b(X)a(Y)$ and by linearity in the argument f. Then

$$\begin{aligned} S(S^{-1}(A)).x &= S((X,Y) \mapsto g(\tilde{X}, A.\tilde{Y})).x \\ &= (X \mapsto g(\tilde{X}, A.\tilde{\tilde{x}}) = g(\tilde{X}, A.x)) \\ &= A.x \end{aligned} \tag{5.26}$$

as we wanted to see.

To summarize the above said, if $\alpha \in \Omega^2 M$, then $S(\alpha)\colon \Omega^{\cdot} M \to \Omega^{\cdot} M$ is the (unique) derivation on the differential forms algebra $\Omega^{\cdot} M$ extending the Lie algebra element in $\mathrm{o}(g)$ corresponding to the two-form α, under the correspondence described above, in equation (5.23).

One can repeat the above for the symmetric algebra: For $a \wedge b \in \Omega^2 M$ and $\alpha \in \mathrm{S}^{\cdot} M$, define $S(a \wedge b).\alpha := (s(a)\iota(b) - s(b)\iota(a)).\alpha$. One can see that S is a derivation, by simply copying the corresponding computations above in the case of S acting on the differential algebra $\Omega^{\cdot} M$, forgetting about the appropriate (-1) signs.

And if $x \in \mathrm{T}^* M$, we get the same formula $S(a \wedge b).x = ag(b,x) - bg(a,x)$ as above. So these facts together show that $S(\alpha)$ is the (unique) derivation on $\mathrm{S}^{\cdot} M$ extending the Lie algebra element in $\mathrm{o}(g)$ corresponding to the two-form α in the correspondence described above.

If you want, head on, to develop a similar formula for S acting on the tensor algebra $\mathrm{T}^{0,\cdot} M$, this does not work, but instead you have to combine the formulas for S acting on $\Omega^{\cdot} M$ and on $\mathrm{S}^{\cdot} M$. For example, for two-cotensors, you write $x \otimes y = \frac{1}{2} x \odot y + \frac{1}{2} x \wedge y$, giving $S(\alpha).(x \otimes y) = \frac{1}{2} S(\alpha).(x \odot y) + \frac{1}{2} S(\alpha).(x \wedge y)$.

For a general cotensor, you end up with a Clebsch-Gordan type of decomposition, corresponding to the decomposition of the tensor product of two irreducible $\mathrm{o}(3)$ spin representations into a direct sum of irreducible modules

$$\mathcal{D}^p \otimes \mathcal{D}^q = \mathcal{D}^{p+q} \oplus \mathcal{D}^{p+q-1} \oplus \ldots \oplus \mathcal{D}^{|p-q|} \tag{5.27}$$

for $p, q \in \frac{1}{2}\mathbb{N}$. — We exclude the details.

5.2 The Hodge Star Operator

We will need to do some computations with the Hodge star operator $*$, which is reviewed in most every book on differential geometry, see for instance Warner [18] or Benn & Tucker [2], but such references only review some of the properties that we need. So it seems appropriate to review the properties needed for our purposes, and also in order to fix notation and making it clear how the computations in this paper are performed.

First, the Hodge star operator $*$ is defined with respect to the metric g; as we, at a later point, will need to let g vary, it is important to keep this fact in mind. Second, even though we only have the (four-dimensional) Lorentz case in mind for the application, the notation is such that it is readily applicable to any manifold with non-degenerate metric g.

The metric volume element is denoted by ω. If $\omega^\mu = dx^\mu$ is any (local) proper orthonormal basis, then $\omega := \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$. One can also for any α in the component of $\Omega^n M$, $n := \dim M$, defined by the orientation, retrieve the metric volume element by the formula

$$\omega := \sqrt{|q(\alpha)|^{-1}}\, \alpha \tag{5.28}$$

which alternatively be taken as the defining formula for ω (useful in performing metric variations).

If now $v \in \Omega^{\cdot} M$ a homogeneous element, then the Hodge star operator value $*v \in \Omega^{\cdot} M$ is defined implicitly to be the element satifying the equation

$$u \wedge *v =: g(u, v)\, \omega \tag{5.29}$$

for all $u \in \Omega^{\cdot} M$ of the same degree as v, $\deg u = \deg v$. It turns out, for instance by equation (5.39) below, that this, for each such v, defines a unique element $*v$, which then is homogeneous of degree $\deg *v = n - \deg v$.

The star operator is then a $\mathcal{C}^\infty M$ linear map $*\colon \Omega^p M \to \Omega^{n-p}$ for $p = 1, \dots, n := \dim M$. One should be careful to note that if $\deg u < \deg *v = n - \deg v$, then the right hand side of the equation (5.29) is always zero, whereas one can find such u for which the left hand is non-zero, so there is a limitation to this formula. In such a case one will have to rely on expansions in terms of the interior multiplication $i(x)$ (cf. equation (5.39) below).

It follows immediately that

$$*1 = \omega, \qquad *\omega = q(\omega) \tag{5.30}$$

Here, $1 \in \Omega^0 M$ is the constant function with value 1, and $q(\omega)$ is the discriminant, equal to $q(\omega^0)q(\omega^1)q(\omega^2)q(\omega^3) = -1$ in the (four-dimensional) Lorentz case. This follows directly from the definition of the star operator $*$: Taking $u = v = 1$, in the defining equation (5.29) above, gives $*1$, and the second equation follows (noting that $*\omega$ is a scalar) from $(*\omega)\omega = \omega \wedge *\omega = g(\omega, \omega)\omega$.

Next, note that for homogeneous elements $u, v \in \Omega^{\cdot} M$ of the same degree,

$$(*u) \wedge v = (-1)^{(n+1)\deg u} g(u, v)\omega \tag{5.31}$$

This follows, since first $(*u) \wedge v = (-1)^{(\deg *u)(\deg v)} v \wedge *u$; then noting that $(\deg *u)(\deg v) = (n - \deg u)(\deg u) \overset{(2)}{\equiv} (n+1)\deg u$, combining with the defining equation (5.29), there results the right hand side of the equation (5.31).

There are the formulas

$$u \wedge *v = v \wedge *u, \qquad (*u) \wedge v = (*v) \wedge u \tag{5.32}$$

which follows, since the metric g is symmetric $g(u,v) = g(v,u)$, directly from the defining equation (5.29), and in the case of the second equation, combined with the equation (5.31) above.

Next, we get

$$*(u \wedge *v) = g(u,v)q(\omega) \tag{5.33}$$

which follows, since $*(u \wedge *v) = *(g(u,v)\omega) = g(u,v) * \omega = g(u,v)q(\omega)$.

Call the elements $u, v \in \Omega^{\cdot} M$ orthogonal (or perpendicular), and write $u \perp v$, if they can be written on the form $u = u_1 \wedge \ldots \wedge u_p$, $v = v_1 \wedge \ldots \wedge v_q$, $u_j, v_k \in \mathrm{T}^* M$, with $g(u_j, v_k) = 0$, for all j, k. Suppose that $\omega = u \wedge v$ with $u \perp v$, so that $\deg u + \deg v = n := \dim M$. Then

$$*u = q(u)v \tag{5.34}$$

This is in fact a useful computational formula.

Proof: Keeping the notation in the orthogonality condition $u \perp v$ from above, we can assume the $u_1, \ldots, u_p, v_1, \ldots, v_q$, $p+q = n$, to form an orthogonal basis, because the equation $\omega = u \wedge v$ together with this orthogonality condition together give $q(\omega) = q(u)q(v)$, forcing $q(u) \neq 0$ and $q(v) \neq 0$, so there is no problem using the Gram-Schmidt process.

Also, if $\alpha \in \Omega^{\cdot} M$ is any differential form of the same degree as u, then it can be written uniquely on the form $\alpha = \lambda u + w$, where $\lambda \in \mathcal{C}^\infty M$ is a scalar function, and $g(w,u) = 0$. Namely, one sees that $\lambda = g(\alpha, u)/q(u)$, $w = \alpha - \lambda u$, (and this implies that $g(u,w) = 0$). Now, writing w as a sum of exterior products with elements from the basis $u_1, \ldots, u_p, v_1, \ldots, v_p$, and some scalar coefficients, it follows that all terms with non-zero coefficients must contain at least one factor among the elements $v_1, \ldots, v_p$; as a consequence, $w \wedge v = 0$.

Then

$$\begin{aligned} \alpha \wedge q(u)v &= \left(\frac{g(\alpha,u)}{q(u)} u + w \right) \wedge q(u)v \\ &= g(\alpha,u) u \wedge v \\ &= g(\alpha,u)\omega \end{aligned}$$

Since this is true for any form α of the same degree as u, the equation (5.34) follows, in view of the defining equation (5.29).

For a homogeneous $u \in \Omega^{\cdot} M$, one has

$$* * u = (-1)^{(n+1)\deg u} q(\omega) u \tag{5.35}$$

with $n := \dim M$, as above. One can also write the factor $(-1)^{(n+1)\deg u} = (-1)^{p(n-p)}$, $p := \deg u$, as $p(n-p) = (\deg u)(n - \deg u) \overset{(2)}{\equiv} (\deg u)(n+1)$.

Proof: By linearity, it is enough to show the formula when u is decomposable, that is $u = u_1 \wedge \ldots \wedge u_p$, with the $u_j \in \mathrm{T}^* M$. We can assume the elements u_j to be orthogonal, and we can complement these elements with elements $v_k \in \mathrm{T}^*M$ to an orthogonal basis in such a way that $\omega = u \wedge v$, with $v = v_1 \wedge \ldots \wedge v_q$, $p + q = n$. In other words, $u \perp v$, and we can apply formula (5.34), giving $*u = q(u)v$.

But we also have $\omega = (-1)^{(\deg u)(\deg v)} v \wedge u = (-1)^{(n+1)\deg u} v \wedge u$. So, again by the equation (5.34), $*v = (-1)^{(n+1)\deg u} q(v)u$. Taken together, this gives $**u = (-1)^{(n+1)\deg u} q(u)q(v)u$, and as $q(u)q(v) = q(\omega)$, the equation (5.35) results.

In the case of the four-dimensional Lorentz metric, as remarked above, $q(\omega) = -1$, and $n + 1 = 5 \overset{(2)}{\equiv} 1$, so the equation (5.35) becomes

$$**u = (-1)^{1+\deg u} u \tag{5.36}$$

For the star operator $*$ acting on two-forms u, the result is then $*^2 = -1$: $\Omega^2 M \to \Omega^2 M$, in other words, as an operator, $*$ defines a complex structure on $\Omega^2 M$. It is this fact that really makes the spin theory presented here. The fact $*^2 = -1$ is being put in the right light only in the context of the Clifford bundle $\mathrm{C}(g)$, where the corresponding formula is $\omega^2 = -1$. So we will have to continue the reviewing of the star operator for some time, and then work out the corresponding Clifford algebra context.

If $\deg u + \deg v = n$, then

$$g(*u, v) = (-1)^{(n+1)p} g(u, *v), \qquad p = \deg v, \text{ or } p = \deg v \tag{5.37}$$

One can also write $(-1)^{(n+1)p} = (-1)^{(\deg u)(\deg v)}$, confer formula (5.36) above. Formula (5.37) follows, since $g(*u, v)\omega = v \wedge u = (-1)^{p(n-p)} u \wedge v = (-1)^{(n+1)p}$ $g(u, *v)\omega$

For any $u, v \in \Omega^{\cdot} M$, one has

$$g(*u, *v) = g(u, v)q(\omega) \tag{5.38}$$

By linearity, it is enough to show this result when u and v are homogeneous, and then it follows by applying, in order, the equations (5.37) and (5.35) above (using $p = \deg v$), as follows $g(*u, *v) = (-1)^{(n+1)p} g(u, **v) = g(u, v)q(\omega)$.

If $u \in \Omega^{\cdot} M$, then

$$*u = i(u)\omega \tag{5.39}$$

This formula is useful, since it makes possible to compute the star operator in terms of the interior multiplication, for which there are better formulas. The formula follows, since for any $a \in \Omega^{\cdot} M$, one has

$$\begin{aligned} g(i(u)\omega, *a) &= g(\omega, u \wedge *a) && \text{by the definition of } i(u) \\ &= g(\omega, \omega) g(u, a) && \text{using } u \wedge *a = g(u, a)\omega \\ &= g(*u, *a) && \text{by formula (5.38)} \end{aligned}$$

The formula (5.39) can be generalized: If $u, v \in \Omega^{\cdot} M$, then

$$*(u \wedge v) = i(v) * u \tag{5.40}$$

Proof: This follows directly from the equation (5.39), as $*(u \wedge v) = i(u \wedge v).\omega = i(v)i(u).\omega = i(v) * u$.

Alternatively, if you do not want to pass over formula (5.39), one can prove it directly as follows: We can assume that u and v are homogeneous; the general case then follows by linearity. Then, for all $a \in \Omega^p M$, where $p := \dim M - \deg u - \deg v$,

$$\begin{aligned} g(i(v) * u, a)\omega &= g(*u, v \wedge a)\omega && \text{by the def of } i(v) \\ &= v \wedge a \wedge * * u && \text{by (5.29)} \\ &= (-1)^{(n+1)\deg u} q(\omega) v \wedge a \wedge u && \text{by (5.35)} \end{aligned}$$

and

$$\begin{aligned} g(*(u \wedge v), a)\omega &= a \wedge * * (u \wedge v) && \text{by (5.29)} \\ &= (-1)^{(n+1)\deg u \wedge v} q(\omega) a \wedge u \wedge v \end{aligned}$$

The last expressions of these two chains of equations agree (as some sign checking reveals), so that $g(i(v) * u, a) = g(*(u \wedge v), a)$ for all $a \in \Omega^{\cdot} M$ in, and so equation (5.40) follows.

If $u, v \in \Omega^{\cdot} M$ are homogeneous elements with $\deg u \leq \deg v$, then

$$*(i(u).v) = (-1)^{\deg u(1+\deg v)} u \wedge *v \tag{5.41}$$

This follows, because for any $a \in \Omega^p M$, $p := \deg a := \deg v - \deg u$, one has

$$\begin{aligned} a \wedge *(i(u).v) &= g(a, i(u).v)\omega \\ &= g(u \wedge a, v)\omega \\ &= u \wedge a \wedge *v \\ &= (-1)^{\deg u \deg a} a \wedge (u \wedge *v) \end{aligned}$$

Now $\deg u \deg a = \deg u(\deg v - \deg u) \overset{(2)}{\equiv} \deg u(1 + \deg v)$, so formula (5.41) follows.

Also, if $u, v \in \Omega^{\cdot} M$ are homogeneous elements, then

$$\begin{aligned} i(u).v &= (-1)^{(n+1)\deg v} q(\omega) * ((*v) \wedge u) \\ &= (-1)^{(n-\deg v)(\deg u - \deg v)} q(\omega) * (u \wedge *v) \end{aligned} \tag{5.42}$$

The first line of this equation is valid also if u is not homogeneous.

This follows since

$$\begin{aligned} i(u).v &= (-1)^{(n+1)\deg v} q(\omega)\, i(u).(**v) && \text{by (5.35)} \\ &= (-1)^{(n+1)\deg v} q(\omega) * ((*v) \wedge u) && \text{by (5.40)} \end{aligned}$$

which gives the first line of the equation. The second line then follows, since $(*v) \wedge u = (-1)^{(n-\deg v)\deg u} u \wedge *v$, and $(n+1)\deg v + (n - \deg v)\deg u \overset{(2)}{\equiv} (n + \deg v)(\deg v - \deg u)$.

If $u, v \in \Omega^{\cdot} M$ are homogeneous elements, then

$$u \wedge v = (-1)^{(n+1)(\deg u + \deg v)} q(\omega) * (i(v). * u) \tag{5.43}$$

Thus follows, since

$$\begin{aligned} u \wedge v &= (-1)^{(n+1)(\deg u + \deg v)} q(\omega) **(u \wedge v) && \text{by (5.35)} \\ &= (-1)^{(n+1)(\deg u + \deg v)} q(\omega) * (i(v). * u) && \text{by (5.40)} \end{aligned}$$

The Hodge star is easy to compute in terms of coordinates: For a sequence of index integers $I = i_1 \dots i_p$, use the shorthand notation

$$\omega^I := \omega^{i_1 \wedge \dots \wedge i_p} := \omega^{i_1} \wedge \dots \wedge \omega^{i_p} \tag{5.44}$$

Then if $J = j_1 \dots j_q$ is any sequence of integers completing I, that is, $p + q$, $I \cup J = \{1, \dots, n\}$, and $I \cap J = \emptyset$, one has

$$*\omega^I = \varepsilon^I{}_J\, \omega^J \qquad \text{no summation over J} \tag{5.45}$$

where $\varepsilon_{k_1 \dots k_n}$ is the symbol defined to be equal to $(-1)^r$, where r is the number of inversions in the sequence $k_1 \dots k_n$; in other words the same thing as the sign of the permutation σ that satisfies $\sigma(l) = k_l$, for $l = 1, \dots, n$.

Written out in coordinates, equation (5.45) reads as any one of the following formulas

$$*\omega^{i_1 \wedge \dots \wedge i_p} = \varepsilon^{i_1 \dots i_p}{}_{j_1 \dots j_q}\, \omega^{j_1 \wedge \dots \wedge j_q} \qquad \text{no summation} \tag{5.46}$$

$$* \,\omega^{i_1} \wedge \dots \wedge \omega^{i_p} = \varepsilon^{i_1 \dots i_p}{}_{j_1 \dots j_q}\; \omega^{j_1} \wedge \dots \wedge \omega^{j_q} \qquad \text{no summation} \tag{5.47}$$

and in both these formulas, as indicated, there must be no summation over the indices $j_1 \dots j_q$.

The proof of formula (5.45) is left to the reader; also see the book by Benn & Tucker [2].

If we apply this to two-forms in a four-dimensional metric, for an even permutation (j, k, l) of $(1, 2, 3)$, we get

$$\begin{cases} *\omega^{0 \wedge j} = -\omega^{k \wedge l} \\ *\omega^{k \wedge l} = \omega^{0 \wedge j} \end{cases} \tag{5.48}$$

since by equation (5.46), we have $*\omega^{0 \wedge j} = \varepsilon^{0j}{}_{kl} \omega^{k \wedge l} = -\varepsilon_{0jkl} \omega^{k \wedge l} = -\omega^{k \wedge l}$, and so on.

5.3 Duality in the Clifford Algebra

We will need to do computations in the Clifford algebra $\mathrm{C}(g)$, using as dualizer $\omega\,\square$, where ω is the metric volume element, and $\square$ is Clifford multiplication, rather than working with the Hodge star operator $*$ acting on the differential algebra $\Omega^{\cdot}$.

The main relation between the Hodge star operator $*$ and the Clifford multiplication is that for $\alpha \in \Omega^{\cdot} = \mathrm{C}(g)$, one has

$$*\alpha = \alpha^{\mathrm{T}} \square\, \omega \tag{5.49}$$

where α^{T} denotes the transpose of α, that is, if $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_p$, then $\alpha^{\mathrm{T}} = \alpha_p \wedge \ldots \wedge \alpha_1$. Instead of writing $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_p$, one can also say that $\alpha = \alpha_1 \square \ldots \square \alpha_p$, with $g(\alpha_j, \alpha_k) = 0$, for $j \neq k$ and $j, k = 1, \ldots, p$.

If α is homogeneous of degree p, then this can also be written

$$\begin{aligned} *\alpha &= (-1)^{\frac{1}{2}p(p-1)} \alpha \square \omega \\ &= (-1)^{\frac{1}{2}p(p+1)} \omega \square \alpha \end{aligned} \tag{5.50}$$

since for degree one elements $x \in \Omega^1$, one has $x \square \omega = -\omega \square x$, and so

$$\alpha \square \omega = (-1)^{\deg \alpha} \omega \square \alpha \tag{5.51}$$

For a proof of formulas (5.49) and (5.50), see the book by Ben & Tucker [2], which gives some other useful formulas, as well.

In the case of a four dimensional Lorentz metric, the equations (5.49) and (5.50) then become, still writing $\deg \alpha = p$,

$$*\alpha = (-1)^{\frac{1}{2}p(p+1)} = \begin{cases} -\omega \square \alpha, & \deg \alpha = 1, 2 \\ \ \ \omega \square \alpha, & \deg \alpha = 0, 3, 4 \end{cases} \tag{5.52}$$

Then, especially in the case of two-forms, for an even permutation (j, k, l) of $(1, 2, 3)$, we get

$$\begin{cases} \omega \square (\omega^0 \square \omega^j) = \omega^k \square \omega^l \\ \omega \square (\omega^k \square \omega^l) = -\omega^0 \square \omega^j \end{cases} \tag{5.53}$$

with a minus sign relative to the corresponding formula (5.48) in the case of the star operator $*$.

Using the shorthand notation $\omega^{0 \square j} := \omega^0 \square \omega^j$, and so on, formula (5.53) can be written

$$\begin{cases} \omega \square \omega^{0 \square j} = \omega^{k \square l} \\ \omega \square \omega^{k \square l} = -\omega^{0 \square j} \end{cases} \tag{5.54}$$

Of course, this formula (and formula (5.53)) can be computed directly $\omega \square \omega^{0 \square j} = (\omega^{0 \square j} \square \omega^{k \square l}) \square \omega^{0 \square j} = \omega^{k \square l} \square (\omega^{0 \square j})^2 = -\omega^{k \square l} (\omega^0)^2 (\omega^j)^2 = \omega^{k \square l}$, and similarly for the second equation of formula (5.54).

6 The Spin Operators; Coordinate Independent Version

We have formerly, in section 2, introduced, using coordinates, some spin operators, moving along the stalk of a four-dimensional Lorentz manifold. In order to resolve certain theoretical problems (as was mentioned in that section, in the paragraph after summary 2.2), we need to do the corresponding coordinate-free version of these spin operators, as well.

Further, this kind of coordinate-free spin operator description will be needed, in order to be able to efficiently tackle the theoretical questions that lie ahead, questions like computing the metric variations, and to use that to obtain a correct generalization of the so called Einstein's equation, to include particles with spin.

So, in order to first describe the spin operators using the Hodge star operator described in section 5, for a two-form $a \wedge b \in \Omega^2$, recall the notation

$$S(a \wedge b) := e(a)i(b) - e(b)i(a) \tag{6.1}$$

also formerly used in equation (5.23); so here $S(a \wedge b) \in \mathrm{o}(g)$.

Written out, acting on an element $x \in M^* = \mathrm{T}^*M$ (in the cotangent bundle), this is

$$\begin{aligned} S(a \wedge b).x &:= (e(a)i(b) - e(b)i(a)).x \\ &= g(b, x)a - g(a, x)b \end{aligned} \tag{6.2}$$

Direct computations then gives, for an even permutation (j, k, l) of $(1, 2, 3)$,

$$\begin{aligned} &S(\omega^{0\wedge j}).x = g(\omega^j, x)\omega^0 - g(\omega^0, x)\omega^j \\ &S(\omega^{0\wedge j})\colon \begin{cases} \omega^0 \to \varepsilon_\sigma \omega^j \\ \omega^j \to \varepsilon_\sigma \omega^0 \end{cases} \qquad \omega^k, \omega^l \to 0 \end{aligned} \tag{6.3}$$

$$\begin{aligned} &S(\omega^{k\wedge l}).x = g(\omega^l, x)\omega^k - g(\omega^k, x)\omega^l \\ &S(\omega^{k\wedge l})\colon \ \ \omega^0, \omega^j \to 0 \qquad \begin{cases} \omega^k \to -\varepsilon_\sigma \omega^l \\ \omega^l \to \varepsilon_\sigma \omega^k \end{cases} \end{aligned} \tag{6.4}$$

where ε_σ is the sign of the spatial part of the metric, that is $\varepsilon_\sigma = \mp$ if the metric g is of type $\pm\mp\mp\mp$.

If we recall, from formula (5.48), that $*\omega^{0\wedge j} = -\omega^{k\wedge l}$, then, combined with equations (6.3) and (6.4), we find that

$$\begin{aligned} &S((1 \pm \mathrm{i}\,*)\omega^{0\wedge j}) = S(\omega^{0\wedge j}) \mp \mathrm{i}\, S(\omega^{k\wedge l}) \\ &S((1 \pm \mathrm{i}\,*)\omega^{0\wedge j})\colon \begin{cases} \omega^0 \to \varepsilon_\sigma \omega^j \\ \omega^j \to \varepsilon_\sigma \omega^0 \end{cases} \quad \begin{cases} \omega^k \to \pm\,\mathrm{i}\, \varepsilon_\sigma \omega^l \\ \omega^l \to \mp\,\mathrm{i}\, \varepsilon_\sigma \omega^k \end{cases} \end{aligned} \tag{6.5}$$

Then, comparing this formula with formula (2.1), we find that

$$S((1 \pm \mathrm{i}\,*)\omega^{0\wedge j}) = \varepsilon_\sigma \sigma^j_\mp \tag{6.6}$$

Now, bearing in mind that $*\alpha = -\omega \mathbin{\scriptstyle\square} \alpha$ when acting on two-forms $\alpha \in \Omega^2$ (cf. equation (5.52)), so that $\omega \mathbin{\scriptstyle\square} \omega^{0\wedge j} = \omega^{k\wedge l} = - * \omega^{0\wedge j}$ (cf. equations (5.54) and (5.48)), equation (6.6) can be written

$$S((1 \pm \mathrm{i}\,\omega) \mathbin{\scriptstyle\square} \omega^{0 \mathbin{\scriptstyle\square} j}) = \varepsilon_\sigma \sigma^j_\pm \tag{6.7}$$

Note here, that since the basis elements ω^μ are orthogonal, Clifford multiplication $\mathbin{\scriptstyle\square}$ and exterior multiplication $\wedge$ can be used interchangeably, like $\omega^{0 \mathbin{\scriptstyle\square} j} = \omega^{0\wedge j}$, $\omega^{k \mathbin{\scriptstyle\square} l} = \omega^{k\wedge l}$, and so forth, when dealing with groups of *distinct* indices.

Of course, formula (6.7), and similar formulas, can be computed entirely within the Clifford algebra environment, and it is instructive to do so, because the computations are often simpler to carry out, than using the star operator $*$.

Then, summarizing the equations (6.6) and (6.7), we get the following:

6.1 Summary. *One has, for the coordinate spin operators $\sigma^j_\pm$, defined by equation (2.1),*

$$\begin{aligned} \sigma^j_\pm &= \varepsilon_\sigma S((1 \pm \mathrm{i}\,\omega) \mathbin{\scriptstyle\square} \omega^{0 \mathbin{\scriptstyle\square} j}) \\ &= \varepsilon_\sigma S((1 \mp \mathrm{i}\,*)\omega^{0\wedge j}) \end{aligned} \tag{6.8}$$

Note that, since $*^2 = -1$ when acting on two-forms, the operators $\frac{1}{2}(1 \pm \mathrm{i}\,*)$, acting on two-forms, are idempotents, that is, each of them is equal to its square; but, since by equation (5.35), one has $* * \alpha = -(-1)^{\deg \alpha}\alpha$, in general, this only holds for the star operator $*$ acting on forms α of even degree.

The situation is different, working with the Clifford bundle $\mathrm{C}(g)$: Clearly $(\omega)^2 = -1$ as an element of $\mathrm{C}(g)$, and so the elements $p_\pm$ defined by

$$p_\pm := \tfrac{1}{2}(1 \pm \mathrm{i}\,\omega) \tag{6.9}$$

are idempotents (that is, $p_\pm^2 = p_\pm$) as elements of $\mathrm{C}(g)$, and therefore also when considered as operators in the situation when whatever any subalgebra of $\mathrm{C}(g)$, which they also are a part of, acts (as algebra) on something else.

In practise, we will take as subalgebra $\mathrm{C}(g)^{\mathrm{even}}$, consisting of the even degree elements of $\mathrm{C}(g)$, acting on the subset $\mathrm{C}(g)^{\mathrm{odd}}$ of $\mathrm{C}(g)$, consisting of the odd degree elements; and clearly $\mathrm{C}(g)^{\mathrm{even}}$ contains the idempotents $p_\pm$. Since the product of an even degree element with an odd degree element is of odd degree, $\mathrm{C}(g)^{\mathrm{odd}}$ becomes a $\mathrm{C}(g)^{\mathrm{even}}$ module.

For later use, we record some relations valid for the idempotents $p_\pm$:

$$\begin{cases} p_\pm^2 = p_\pm \\ p_+ \mathbin{\scriptstyle\square} p_- = 0 \end{cases} \qquad \begin{cases} p_+ + p_- = 1 \\ p_+ - p_- = \mathrm{i}\,\omega \end{cases} \tag{6.10}$$

$$p_\pm \mathbin{\scriptstyle\square} \alpha = \begin{cases} \alpha \mathbin{\scriptstyle\square} p_\pm, & \deg \alpha \overset{(2)}{\equiv} 0 \\ \alpha \mathbin{\scriptstyle\square} p_\mp, & \deg \alpha \overset{(2)}{\equiv} 1 \end{cases} \tag{6.11}$$

which are straightforward to compute. The relations (6.10) are those that really are making the splittings described below. The odd/even degree asymmetry of the relations (6.11), forces us to go down to the even sub Clifford algebra $\mathrm{C}(g)^{\mathrm{even}}$.

The effect of the presence of the idempotents $p_\pm$ is that the Clifford bundle $\mathrm{C}(g)$ is being split up in two parts

$$\begin{aligned}\mathrm{C}(g)_\pm &:= \mathrm{C}(g) \circ p_\pm \\ &= \{\alpha \in \mathrm{C}(g) \mid \alpha \circ p_\pm = \alpha\}\end{aligned} \tag{6.12}$$

into a direct sum decomposition

$$\mathrm{C}(g) = \mathrm{C}(g)_+ \oplus \mathrm{C}(g)_- \tag{6.13}$$

that is $\mathrm{C}(g) = \mathrm{C}(g)_+ + \mathrm{C}(g)_-$ and $\mathrm{C}(g)_+ \cap \mathrm{C}(g)_- = 0$. The second equality of equation (6.12) is shown by direct computation, using the relations in equation (6.10), as follows: If $\alpha \in \mathrm{C}(g)_\pm$, then $\alpha = \beta \circ p_\pm$, for some $\beta \in \mathrm{C}(g)$, and then $\alpha \circ p_\pm = \beta \circ p_\pm^2 = \beta \circ p_\pm = \alpha$, showing the inclusion $\subset$. On the other hand, if $\alpha = \alpha \circ p_\pm$, then clearly $\alpha \in \mathrm{C}(g) \circ p_\pm$, which shows the other inclusion $\supset$.

Because of this splitting, of equation (6.13), various subparts of the Clifford algebra $\mathrm{C}(g)$ also split into $\pm$ parts. To preview the results to come below, the total outcome is that the $\pm$ signs correspond to the particles/antiparticles structure.

So, in section 5, via equation (5.23), we identified the (real) Lie algebra $\mathrm{o}(g)$ with the set of differential two-forms Ω^2, and since we also in section 4 identified $\Omega^\cdot = \mathrm{C}(g)$, the splitting, as in equation (6.13), caused by the idempotents $p_\pm$, also splits up the Lie algebra $\mathrm{o}(g)$ into a direct sum decomposition

$$\mathrm{o}(g) = \mathrm{o}(g)_+ \oplus \mathrm{o}(g)_- \tag{6.14}$$

into two sub Lie algebras $\mathrm{o}(g)_\pm$. Note here, because the idempotents $p_\pm$ contain the complex imaginary unit $\mathrm{i} = \sqrt{-1}$, the splitting in equation (6.14) will, when written out in coordinates, involve complex numbers; however, the vectorspaces $\mathrm{o}(g)_\pm$ are still regarded as real vectorspaces.

As we shall see below in section 7, this direct sum decomposition in equation (6.14) is the same, in a suitable coordinate context, as the classical decomposition that results in the classification of spin into types (p, q), where $p, q \in \frac{1}{2}\mathbb{N}$.

One physical interpretation of this result (equation (6.14)), is that the structure of particles/antiparticles do exist in a general relativistic context, not dependent on observed data (that is, the presence of a predesignated time coordinate). Further, particles that are self-dual[38] (that is, particles that are their

[38]Since this is a statement of merely a theory attempting to describe the real physical universe, and where what you actually detect in this physical universe (through measurements, experiments, etc.) is what really counts, you should rightly at this point insert speculations about what the correct description ought to be, in the case one would happen to actually find, say, a self-dual particle with non-integral total spin. However, no such particle has yet been detected, and the interpretation given here seems the most natural.

own antiparticles, such as the photons) must be of bispin type (p,p), since if a particle is of bispin type (p,q), its antiparticle is of bispin type (q,p). Since $p \in \frac{1}{2}\mathbb{N}$, and the total spin of a particle with bispin (p,q) is $p+q$, such self-dual particles must be of total (non-negative) integral spin.

In particular, neutrinos[39], which are of total spin $\frac{1}{2}$, and therefore must be of bi-spin $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$, cannot, by this reasoning, be self-dual.

Just as the real Lie algebra $\mathrm{o}(g)$ is being split up into $\pm$ parts, as in equation (6.14), so are the even and odd degree parts of the Clifford bundle being $\mathrm{C}(g)$ split up into $\pm$ parts, but then, in addition, we have to take into account the asymmetries of equation (6.11). If an element $\alpha \in \mathrm{C}(g)^{\mathrm{even}}$, and $\alpha = \alpha \centerdot p_\pm$, then, by equation (6.11), this is equivalent to that $\alpha = p_\pm \centerdot \alpha$, whereas if $\alpha \in \mathrm{C}(g)^{\mathrm{odd}}$ and $\alpha = \alpha \centerdot p_\pm$, then, again by equation (6.11), this is equivalent to that $\alpha = p_\mp \centerdot \alpha$. From this we get the following

$$(6.15)\begin{cases} \mathrm{C}(g)^{\mathrm{even}} \centerdot p_\pm = p_\pm \centerdot \mathrm{C}(g)^{\mathrm{even}} \\ \qquad = \{\alpha \in \mathrm{C}(g)^{\mathrm{even}} \mid \alpha = \alpha \centerdot p_\pm\} = \{\alpha \in \mathrm{C}(g)^{\mathrm{even}} \mid \alpha = p_\pm \centerdot \alpha\} \end{cases}$$

$$(6.16)\begin{cases} \mathrm{C}(g)^{\mathrm{odd}} \centerdot p_\pm = p_\mp \centerdot \mathrm{C}(g)^{\mathrm{odd}} \\ \qquad = \{\alpha \in \mathrm{C}(g)^{\mathrm{odd}} \mid \alpha = \alpha \centerdot p_\pm\} = \{\alpha \in \mathrm{C}(g)^{\mathrm{odd}} \mid \alpha = p_\mp \centerdot \alpha\} \end{cases}$$

In other words, when we use the notation $\mathrm{C}(g)^{\mathrm{even}}_\pm$, we need not worry about whether the notation involvs multiplication with the idempotents $p_\pm$ from the left, or from the right. But the situation is different with $\mathrm{C}(g)^{\mathrm{odd}}_\pm$; we introduce the notation

$$(6.17) \qquad {}_\pm\mathrm{C}(g)^{\mathrm{odd}} := p_\pm \centerdot \mathrm{C}(g)^{\mathrm{odd}}$$

so that (part of) equation (6.16) reads

$$(6.18) \qquad \mathrm{C}(g)^{\mathrm{odd}}_\pm = {}_\mp\mathrm{C}(g)^{\mathrm{odd}}$$

Recall that we are in a situation of mixed rationality, with some real objects acting on some complex objects: Generally $\mathrm{C}(g)^{\mathrm{even}}_\pm$ will be a real vectorbundle, the projection of the real vectorbundle $\mathrm{C}(g)^{\mathrm{even}}$ under the idempotents with complex coefficents $p_\pm$; in a physics context, $\mathrm{C}(g)^{\mathrm{even}}$ will contain observables. On the other hand, ${}_\pm\mathrm{C}(g)^{\mathrm{odd}} = \mathrm{C}(g)^{\mathrm{odd}}_\mp$ will be a complex vectorbundle; in a physics context the elements of $\mathrm{C}(g)^{\mathrm{odd}}$ will be statistical fields. For this reason, the notations $\mathrm{C}(g)^{\mathrm{odd}}_{\mathbb{C}}$, $\mathrm{C}(g)^{\mathrm{odd}}_{\pm,\mathbb{C}}$ and ${}_\pm\mathrm{C}(g)^{\mathrm{odd}}_{\mathbb{C}}$ would be more stringent, but in order to avoid too cumbersome notation, those extra indices $\mathbb{C}$ are being dropped.

One may also remark that, for any element $\alpha \in \mathrm{C}(g)$, the relation $\alpha \centerdot p_\pm = p_\pm \centerdot \alpha$ is equivalent to that $\alpha \centerdot \omega = \omega \centerdot \alpha$, which in its turn is equivalent to that

[39]The current state of this matter, is that no-one for sure has been able to determine whether the neutrinos are self-dual or not. However, if you assume the preservation of the lepton number, neutrinos and anti-neutrinos have lepton numbers with opposite signs.

α is of even degree. Similarly, the relation $\alpha \circ p_\pm = p_\mp \circ \alpha$ is equivalent to that $\alpha \circ \omega = -\omega \circ \alpha$, which in its turn is equivalent to that α is of odd degree. (This reasoning only uses formula (5.51).) Therefore we can write

$$\mathrm{C}(g)^{\mathrm{even}}_\pm = \{\alpha \in \mathrm{C}(g) \mid \alpha = \alpha \circ p_\pm = p_\pm \circ \alpha\} \tag{6.19}$$

$$\mathrm{C}(g)^{\mathrm{odd}}_\pm = \{\alpha \in \mathrm{C}(g) \mid \alpha = \alpha \circ p_\pm = p_\mp \circ \alpha\} \tag{6.20}$$

Note, that because of the relations of equation (6.11), the subsets $\mathrm{C}(g)^{\mathrm{even}}_\pm$ are clearly algebras, since they are left $\mathrm{C}(g)^{\mathrm{even}}$ ideals. If we multiply two elements $\alpha, \beta \in \mathrm{C}(g)^{\mathrm{even}}_\pm$, then we get a formula of the type $\alpha \circ \beta = \alpha \circ p_\pm \circ \beta \circ p_\pm = \alpha \circ \beta \circ p_\pm^2 = \alpha \circ \beta \circ p_\pm$.

If we multiply an element $\alpha \in \mathrm{C}(g)^{\mathrm{even}}_\pm$ with an element $\beta \in \mathrm{C}(g)^{\mathrm{odd}}_\pm$, then by the relations in formula (6.11), and also by formula (6.10), we get $\alpha \circ \beta = \alpha \circ p_\pm \circ \beta \circ p_\pm = \alpha \circ p_\pm \circ p_\mp \circ \beta = 0$. Because of this, the only interesting $\mathrm{C}(g)^{\mathrm{even}}$ module action on $\mathrm{C}(g)^{\mathrm{odd}}$ is the one of $\mathrm{C}(g)_\pm$ acting on ${}_\pm\mathrm{C}(g)^{\mathrm{odd}}$; below, in equations (6.69) and (6.70), we shall see that this action may be identified with the spin operator action $\sigma^j_\pm$ onto the vectorbundle $V_{\mathbb{C}} = M^* \otimes_{\mathbb{R}} \mathbb{C}$, which thus gives a spin representation of bispin type $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ (corresponding to the sign $\pm$), and of degeneracy two.

Similarly, if we multiply two elements $\alpha, \beta \in \mathrm{C}(g)^{\mathrm{odd}}_\pm$, then we find $\alpha \circ \beta = \alpha \circ p_\pm \circ \beta \circ p_\pm = \alpha \circ p_\pm \circ p_\mp \circ \beta = 0$ (by formulas (6.11) and (6.10)). Therefore the only non-zero multiplication between different combinations of the subsets ${}_\pm\mathrm{C}(g)^{\mathrm{odd}} = \mathrm{C}(g)^{\mathrm{odd}}_\mp$ of $\mathrm{C}(g)$ are

$$\begin{cases} {}_\pm\mathrm{C}(g)^{\mathrm{odd}} \otimes {}_\mp\mathrm{C}(g)^{\mathrm{odd}} \longrightarrow \mathrm{C}(g)^{\mathrm{even}}_\pm \\ \qquad\qquad\qquad \alpha \otimes \beta \longrightarrow \alpha \circ \beta \end{cases} \tag{6.21}$$

since, if $\alpha \in {}_\pm\mathrm{C}(g)^{\mathrm{odd}}$ and $\beta \in {}_\mp\mathrm{C}(g)^{\mathrm{odd}}$, then (using formulas (6.11) and (6.10)) $\alpha \circ \beta = p_\pm \circ \alpha \circ p_\mp \circ \beta = \alpha \circ p_\mp \circ p_\mp \circ \beta = \alpha \circ p_\mp \circ \beta = \alpha \circ \beta \circ p_\pm$, so that $\alpha \circ \beta \in \mathrm{C}(g)^{\mathrm{even}}_\pm$.

We will need to return to the multiplications in formula (6.21), below in section 7, when describing the isomorphism $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) \cong (\frac{1}{2}, \frac{1}{2})$; among other things, we will find that this isomorhism depends on a timelike vector. The situation is thus more complicated than in typical quantum mechanics, where one merely assumes this isomorphism (one gives a so called axiomatic description).

Let us return, in order to complete it, to the coordinate free description of the spin operators $\sigma^j_\pm$ in summary 6.1 above.

If, in the equation

$$\sigma^j_\pm = \varepsilon_\sigma S((1 \pm \mathrm{i}\,\omega) \circ \omega^{0 \circ j})$$

of summary 6.1, we let the right hand side vary over the $\mathbb{R}$ linear span over the two-forms of the form $\omega^{0 \circ j} = \omega^{0 \wedge j}$, the left hand side of this equation shows that we cover the $\mathbb{R}$ linear span of the operators $\sigma^j_\pm$ exactly once.

However, the $\mathbb{R}$ linear span of the two-forms $\omega^{0\wedge j}$ is highly dependent on the time coordinate $\omega^0 = dt$, so this does not show (and, which, in fact, is not true) that the $\mathbb{R}$ linear span of the operators $\sigma^j_\pm$ is coordinate independent.

But, by equation (5.54), we also have, for an even permutation (j, k, l) of $(1, 2, 3)$, the equalities $\omega \square \omega^{0 \square j} = \omega^{k \square l}$ and $\omega \square \omega^{k \square l} = -\omega^{0 \square j}$, so that $(1 \pm \mathrm{i}\,\omega) \square \omega^{0 \square j} = \omega^{0 \square j} \pm \mathrm{i}\,\omega^{k \square l}$, and therefore

$$\begin{aligned}
\pm\,\mathrm{i}(1 \pm \mathrm{i}\,\omega) \square \omega^{k \square l} &= \pm\,\mathrm{i}(\omega^{k \square l} \mp \mathrm{i}\,\omega^{0 \square j}) \\
&= \pm\,\mathrm{i}\,\omega^{k \square l} + \omega^{0 \square j} \\
&= (1 \pm \mathrm{i}\,\omega) \square \omega^{0 \square j}
\end{aligned}$$

In other words,

$$\tag{6.22} \begin{cases} \pm\,\mathrm{i}(1 \pm \mathrm{i}\,\omega) \square \omega^{k \square l} = (1 \pm \mathrm{i}\,\omega) \square \omega^{0 \square j} \\ \qquad\quad \pm\,\mathrm{i}\,p_\pm \square \omega^{k \square l} = p_\pm \square \omega^{0 \square j} \end{cases}$$

Applying this formula (6.22), we find that the set $\{S((1 \pm \mathrm{i}\,\omega) \square F) \mid F \in \Omega^2_{\mathbb{R}}\}$, where F thus ranges over the set of *real*[40] two-forms $\Omega^2_{\mathbb{R}}$, and which clearly is a coordinate independent set, is the *complex* span of the operators $\sigma^j_\pm$. We display this result for future reference:

$$\tag{6.23} (\sigma^j_\pm)_{\mathbb{C}} = \{S((1 \pm \mathrm{i}\,\omega) \square F) \mid F \in \Omega^2_{\mathbb{R}}\}$$

Clearly, this is as far as we can get with respect to a coordinate independent description of a suitable span of the operators $\sigma^j_\pm$.

From the point of view of physics, so is the fact that only the complex span (and not the real span) of the operators $\sigma^j_\pm$ is coordinate independent not so surprising, since in quantum mechanics we will study the action of the operators on complex vectorspaces (like the vectorspaces $\mathrm{W}^\pm_\gamma$, defined in the statement 3.3 above, and which are given a coordinate independent description below), and we can never expect to be able to universally fix phase factors to be real.

It is now possible to make use of the equation (6.23), to give coordinate independent descriptions of various invariances of the vectorspaces $\mathrm{W}^\pm_\gamma$ defined in the statement 3.3 above. But first, we need to introduce some notation.

Let us define the coordinate free version of the operator J^j, defined in equation (2.26). If τ is any normalized (cotensor) time-coordinate, in practise being taken future pointing, for any cotensor u orthogonal to τ, write

$$\tag{6.24} J^u := \varepsilon_\tau S(\omega \square (\tau \wedge u))$$

where ε_τ is the sign of the temporal (time) part of the metric g, that is, $\varepsilon_\tau = \pm$ if g is of type $\pm\mp\mp\mp$; thus $\varepsilon_\tau \varepsilon_\sigma = -1$.

[40]By $\Omega^{\cdot}$ is always (in this paper) meant the set of real differential forms; the notation $\Omega^{\cdot}_{\mathbb{R}}$ is used only to emphasize that the forms under consideration are indeed real.

If we set $\tau := \omega^0$, and $u := \omega^j$, then direct computations show that we get the operator J^j defined in equation (2.26), as follows: Using equation (5.54), we have $J^{\omega^j}.x := \varepsilon_\tau S(\omega \circ \omega^{0\wedge j}).x = \varepsilon_\tau S(\omega^{k\wedge l}).x = \varepsilon_\tau(e(\omega^k)i(\omega^l) - e(\omega^l)i(\omega^k)).x = \varepsilon_\tau(g(\omega^l, x)\omega^k - g(\omega^k, x)\omega^l)$. Then, by simply plugging in the values $x = \omega^\mu$ into this formula, the result follows; for instance, $J^{\omega^j}.\omega^k = \varepsilon_\tau(-q(\omega^k)\omega^l) = -\varepsilon_\tau\varepsilon_\sigma\omega^l = \omega^l$, and continuing like this, reproduces the defining formula (2.26) of the operator J^j.

Also note that if we write

$$\sigma_\pm^u := \varepsilon_\sigma S((1 \pm \mathrm{i}\,\omega) \circ (\tau \wedge u)) \tag{6.25}$$

so that $\sigma_\pm^{\omega^j} = \sigma_\pm^j$, then we have

$$J^u = \frac{\mathrm{i}}{2}(\sigma_+^u - \sigma_-^u) \tag{6.26}$$

just as in the case of the operators J^j in the proof of formula (2.27).

We can also describe the action of the operator J^u in coordinates using the (three dimensional) cross product $\times$, as follows:

$$J^u.v = \begin{cases} 0, & v \parallel \tau, u \\ u \times v, & v \perp \tau, u \end{cases} \tag{6.27}$$

Here $v \parallel \tau, u$ means that v is parallel to both τ and u, and it is the same thing as saying that $u \in (\tau, u)_{\mathbb{R}}$, the real vectorspace generated by τ and u; and $v \perp \tau, u$ means that u is perpendicular to both τ and u, that is $g(v, \tau) = g(v, u) = 0$. Since both $q(\tau)$ and $q(u)$ are non-zero, the vectorspace consisting of all elements v perpendicular to τ and u is two dimensional, just as $(\tau, u)_{\mathbb{R}}$ is, and the intersection of these two vectorspaces is zero; so therefore formula (6.27) gives a full description of the operator J^u.

In order to be able to compute the cross product $u \times v$ of formula (6.27), some interpretation is needed, following the lines of appendix A. In practise, in an explicit computation, all that one needs to do is to express in coordinates $u = u_\mu$ and $v = v_\mu$ in an orthonormal basis ω^μ in which $\tau = \omega^0$, and then use the usual coordinate formula for $u \times v$. In order to see that this works, use the formula $(u \times v) \cdot \tilde{\boldsymbol{\omega}}$ (cf. the remark after formula (A.4)) in order to compute $\omega^j \times \omega^m$ for $m = j, k, l$; one then finds that this agrees with the values $J^{\omega^j}.\omega^m$ (and which are the values indicated in formula (2.26)).

Taking these conventions into account, we can simply write equation (6.27)

$$J^u.v = \begin{cases} 0, & v \parallel \tau \\ u \times v, & v \perp \tau \end{cases} \tag{6.28}$$

or simply $J^u.v = u \times v$, keeping in mind that the time component of v gives zero as output in $J^u.v$.

So we see, that in quantum mechanics, $\hbar J$, where $\hbar$ denotes the Planck constant, will simply be the angular momentum operator. Formula (6.26) above then can be written

$$\hbar J^u = \frac{\mathrm{i}\,\hbar}{2}\sigma^u_+ - \frac{\mathrm{i}\,\hbar}{2}\sigma^u_- \tag{6.29}$$

and this gives a Lorentzian explanation[41] of why spins are being halved. For instance, a field of bispin type $(\frac{1}{2}, 0)$ will give zero as result when acted upon by the spin operators σ^u_-, so equation (6.26) then reduces to the usual spin $\frac{1}{2}$ formula; a similar thing happens for fields of bispin type $(0, \frac{1}{2})$, in which case it is the operator σ^u_+ that gives output zero as result, when acting on such a field.

Let us then describe the above promised invariances, in coordinate independent description:

6.2 Proposition. *Let $\gamma \in \mathcal{L}$ be a (non-zero) element on the (real) lightcone. Then the following is true:*

i. One has

$$\mathrm{W}^{\pm}_{\gamma} = \{S((1 \pm \mathrm{i}\,\omega) \circ F).\gamma \mid F \in \Omega^2_{\mathbb{R}}\} \tag{6.30}$$

that is, $\mathrm{W}^{\pm}_{\gamma}$ is the orbit of γ under the action of the spin operators $S((1\pm\mathrm{i}\,\omega) \circ F)$, where F ranges over all real two-forms.

Since $\mathrm{W}^{\pm}_{\gamma}$ is a (two-dimensional) complex vectorspace, one then also has

$$\mathrm{W}^{\pm}_{\gamma} = \{S((1 \pm \mathrm{i}\,\omega) \circ \alpha).\gamma \mid \alpha \in \Omega^2_{\mathbb{C}}\} \tag{6.31}$$

for complex *two-forms $\alpha \in \Omega^2_C$, and, for any such α, also*

$$S((1 \pm \mathrm{i}\,\omega) \circ \alpha).\mathrm{W}^{\pm}_{\gamma} \subset \mathrm{W}^{\pm}_{\gamma} \tag{6.32}$$

ii. The set of operators $S((1 \pm \mathrm{i}\,\omega) \circ F)$, for $F \in \Omega^2_{\mathbb{R}}$, acts transitively on $\mathcal{W}^{\pm}$, the latter[42] being the set whose elements are the vectorspaces $\mathrm{W}^{\pm}_{\gamma}$, for (non-zero) $\gamma \in \mathcal{L}$.

Using the definitions of $\sigma^u_{\pm}$ in equation (6.25) and J^u in equation (6.24), one has

$$\exp(\mathrm{i}\,\theta\sigma^u_{\mp}).\mathrm{W}^{\pm}_{\gamma} = \mathrm{W}^{\pm}_{\gamma'}\,, \quad \text{where } \gamma' = \exp(\mp 2\theta J^u).\gamma \tag{6.33}$$

iii. If $u \perp \tau$ and $u \neq 0$, then

$$\sigma^u_{\pm}.\mathrm{W}^{\pm}_{\gamma} = \mathrm{W}^{\pm}_{\gamma} \tag{6.34}$$

[41] One could say that this is being made possible by a rigorous application of the general relativistic dictionary, separating time coordinates that belong to an observer, from time coordinates that belong to a coordinate system.

[42] Cf. the text in the beginning of section 3, up to proposition 3.1.

Further, the set

$$\{\sigma_{\pm}^{u}.\gamma \mid u \in \mathrm{T}^* M\} \tag{6.35}$$

spans the vectorspace over the complex numbers $\mathbb{C}$, as does the set

$$\{\sigma_{\pm}^{j}.\gamma \mid j = 1, 2, 3\} \tag{6.36}$$

In addition, the set

$$\{\sigma_{\pm}^{j}.\gamma, \sigma_{\pm}^{2}\sigma_{\pm}^{1}.\gamma\} \tag{6.37}$$

spans $\mathrm{W}_{\gamma}^{\pm}$ over the real numbers $\mathbb{R}$.

Proof: Most of this proposition is a coordinate independent variation of facts that we already know about; one simply writes out the statements in coordinates, and sees that it reduces to the facts known in a coordinate setting.

One exception is equation (6.34), which requires some extra effort.

So, in order to prove *i*, we know from proposition 3.2 that there is a restricted orthonormal basis ω^{μ} in which $\gamma = dt + dz$ and $\mathrm{W}_{\gamma}^{\pm} = (dt + dz, dx \pm \mathrm{i}\, dy)_{\mathbb{C}}$; then it reduces to a question of the linear span of the operators $\sigma_{\pm}^{j}$ acting on γ and $\mathrm{W}_{\gamma}^{\pm}$. From formula (6.22) above, we already know that $S((1 \pm \mathrm{i}) \cdot F)$, with F ranging over the set of all real two-forms, reduces to the complex span of the operators $\sigma_{\pm}^{j}$. We then already noted, in the paragraph preceeding summary 2.2, that the operators $\sigma_{\pm}^{j}$ leaves the isotropic vectorspaces invariant, and therefore so does any operator in their complex span, so this gives equation (6.32).

Now, equations (2.9) and (2.11) show that

$$\begin{cases} \sigma_{\pm}^{3}.\gamma = \gamma \\ \sigma_{\pm}^{1}.\gamma = dx \pm \mathrm{i}\, du \end{cases} \tag{6.38}$$

and so it follows that

$$\mathrm{W}_{\gamma}^{\pm} = (\sigma_{\pm}^{3}.\gamma, \sigma_{\pm}^{1}.\gamma)_{\mathbb{C}}$$

and this shows equalities (6.30) and (6.31) then, and similarly, we get the equations (6.35) and (6.36).

In order to get equation (6.37), we note that from formulas (2.13) and (2.11), we get

$$\begin{cases} \sigma_{\pm}^{2}.\gamma = \mp \mathrm{i}(dx \pm \mathrm{i}\, dy) \\ \sigma_{\pm}^{2}\sigma_{\pm}^{1}.\gamma = \pm \mathrm{i}\, \gamma \end{cases} \tag{6.39}$$

By combining the equations (6.38) and (6.39) we then get the equality (6.37).

Equation (6.33) is just a restatement of equation (2.29), and therefore also the transitivity in *ii* follows, of course.

There remains formula (6.34) to be shown. Just as in the proof of proposition 3.2, we can find a restricted orthonormal basis ω^μ in which $\tau = dt$ and $\mathrm{W}_\gamma^\pm = (dt+dz, dx \pm \mathrm{i}\, dy)_\mathbb{C}$; the question then reduces considering the coordinate spinoperators $\sigma_\pm^j$ of section 2.

The strategy is now to show that $(\sigma_\pm^u)^2$ is a homothety, that is, a non-zero scalar times the identity operator; if this is shown, it follows that $\sigma_\pm^u$ is invertible, and formula (6.34) then follows.

Expanding $u = u_j \omega^j$ into coordinates (and $u_0 = 0$ since $u \perp \tau$) gives, using the relations in the statement 2.1,

$$
\begin{aligned}
(\sigma_\pm^u)^2 &= (\sum_j u_j \sigma_\pm^j)^2 = \sum_j (u_j)^2 (\sigma_\pm^j)^2 + \sum_{j,k} u_j u_k \sigma_\pm^j \sigma_\pm^k \\
&= (\sum_j (u_j)^2) \cdot 1 = (\mathbf{u} \cdot \mathbf{u}) \cdot 1
\end{aligned}
\tag{6.40}
$$

a homothety, as we wanted to show.

It is possible to leap to a wholly coordinate free version of the spin operators $\sigma_\pm^u$ by simply writing, for a two-form $F \in \Omega^2$,

$$
\sigma_\pm^F := \varepsilon_\sigma S((1 \pm \mathrm{i}\,\omega) \mathbin{\scriptstyle\square} F)
\tag{6.41}
$$

Then, by the summary 6.1, if τ is a time-like, future-pointing normalized cotensor, one has

$$
\sigma_\pm^u = \sigma_\pm^{\tau \wedge u}
\tag{6.42}
$$

In the application, the two-form F will simply be the electromagnetic two-form (or the Lorentz version of an infinitesimal angular momentum deformation).

6.1 The Representations of Spin and Anti-spin One-half

We are now in position to define the mathematically intrinsic coordinate independent version of the representations of bispin types $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

The wording *mathematically intrinsic* here means that the operators and what they act upon themselves are tensors. This is opposed to an axiomatic description where you simply assume the existence of a representation and hooks it onto the manifold, say through a group bundle construction, without specifying sufficient data about how this is done; such constructions are called *mathematically extrinsic*. The main point in our context is that the mathematically intrinsic construction will be needed for the computation of the generalization of the so called Einstein equation in general relativity, to include particles with (physically instrinsic) spin.

In general, if a representation is of bispin type (p, q), with $p, q \in \frac{1}{2}\mathbb{N}$, then it will be said to be of spin p and anti-spin q (and of total spin $p + q$). The representation of bispin type $(\frac{1}{2}, 0)$ can thus be said to be of spin one half, and the representation of bispin type $(0, \frac{1}{2})$ can thus be said to be of anti-spin one-half. This corresponds to the situation of particles/anti-particles in physics, but the assignment is of course completely arbitrarily, in the sense that what is arbirarily termed, say, a spin one half anti-particle, could have either spin one half or anti-spin one half, depending on how you set up your terminology.

Now define the representations $\rho_\pm \colon \mathrm{o}(g) \times V^\pm \to V^\pm$ of the Lie algebra $\mathrm{o}(g)$ by arbitrarily setting the action[43] of $\mathrm{o}(g)_\mp$ equal to zero in the $\mathrm{o}(g)$ action on the[44] complex vectorspace $V_{a,\mathbb{C}} = M_a^* \otimes_\mathbb{R} \mathbb{C}$ at the point $a \in M$, smoothly varying[45] with the point a then. In other words, forgetting the representation structure, the vector bundle $V^\pm$ is identical with $V_\mathbb{C}$, and if $F \in \Omega^2$ and $x \in V^\pm$ then

$$\text{(6.43)} \qquad \begin{cases} \rho_\pm((1 \mp \mathrm{i}\,\omega) \circ F).x := 0 \\ \rho_\pm(F).x := S((1 \pm \mathrm{i}\,\omega) \circ F).x \end{cases}$$

and the action of an element in the Lie algebra $\mathrm{o}(g)$ on $V^\pm$ is then gotten by the usual[46] identification $\mathrm{o}(g) = \Omega^2$ then.

In view of equation (6.29), one might think that one should put in a factor $\frac{1}{2}$ on the right hand side of the second equation of the formula (6.43); however, one does not land on the Clifford multiplication then (cf. equation (6.68) below). This is also the reason for excluding the factor ε_σ here (cf. summary 6.1 and equation (6.41); using the notation of equation (6.41), we thus have

$$\text{(6.44)} \qquad \rho_\pm = \varepsilon_\sigma \sigma_\pm^F$$

We already know, from equation (6.14) above, that equation (6.43) is well defined but we will see below[47] that it is possible to extricate these representations by first identifying $\mathrm{o}(g)$ via formula (5.23) with $\mathrm{C}(g)^2 = \Omega^2$, and then having $\mathrm{C}(g)^{\mathrm{even}}$ acting by the usual Clifford multiplication on the modules $\mathrm{C}(g)_\pm^{\mathrm{odd}}$ of equations (6.16) and (6.20), the latter which can be identified with the modules $V^\pm$; these identifications are done via formulas (6.51) and (6.54).

Once we have the representations the total tensorspin space can be built, being simply the tensor algebra bundle

$$\text{(6.45)} \qquad \mathrm{T}(V^+ \oplus V^-) \cong \mathrm{T}V^+ \otimes \mathrm{T}V^-$$

There results a tensor bigrading on this cotensor algebra

$$\text{(6.46)} \qquad \begin{aligned} \mathrm{T}^{(p,q)}(V^+ \oplus V^-) &= V^{+\otimes p} \otimes V^{-\otimes q} \\ &= \underbrace{V^+ \otimes \ldots \otimes V^+}_{p \text{ times}} \otimes \underbrace{V^- \otimes \ldots \otimes V^-}_{q \text{ times}} \end{aligned}$$

[43] See equation (6.14) above.

[44] See beginning of section 2.

[45] So, strictly speaking, here $V_\mathbb{C} = M^* \otimes_\mathbb{R} \mathbb{C}$ is the complexified cotangent bundle, as opposed to the case in sections 2 and 3, where $V_\mathbb{C}$ merely denotes a vectorspace, the complexified stalk $M_a^* \otimes_\mathbb{R} \mathbb{C}$ at a point $a \in M$.

[46] See formula (5.23), and the text following it.

[47] See formula (6.69) below.

and note that the bigrading is written with parenthesises $T^{(p,q)}$ in order to not confuse these solely cotensors with the tensor/cotensor bigrading $\mathrm{T}^{p,q}$ discussed in section 4.

The spin bigrading $\mathrm{T}^{(p,q)}$ is related to the bigrading of complex manifolds into holomorhic and antiholomorphic tensors, but with the difference that the situation here is more complicated: Each choice of a direct decomposition pair $V_{\mathbb{C}} = \mathrm{W}^+_\gamma \otimes \mathrm{W}^{+,\mathrm{cx}}_{\gamma'}$, with W^+_γ and $\mathrm{W}^{+,\mathrm{cx}}_{\gamma'}$ being maximal isotropic vectorspaces (cf. the statements 3.8 and 3.6) corresponds (in the case of a positive definite metric) to a choice of a complex structure. In the Lorentz case, compared to case of the positive definite metric complex structures, the situation corresponds to describing all such possible complex structures simultaneously, relating to the dependency of the lines γ, γ' on the real light-cone $\mathcal{L}$.

Returning to the spin bigrading, as in equation (6.46), also note that the $\mathrm{o}(g)$ modules $\mathrm{T}^{(2p,2q)}$, where, for the purpose of this discussion, $p, q \in \frac{1}{2}\mathbb{N}$, are not irreducible modules, and so therefore they are not of bispin type (p,q). First there is a degeneracy two for each of the vectorspaces $V^\pm$ and in order to get an irreducible module the vector space $V^\pm$ has to be reduced to one of the vectorspaces $\mathrm{W}^\pm_\gamma$ for a choice of a line γ on[48] the real light-cone $\mathcal{L}$ (and then any choice of a line γ' on $\mathcal{L}$ distinct from γ gives a complementary irreducible module $\mathrm{W}^\pm_{\gamma'}$ so that $V^\pm = \mathrm{W}^\pm_\gamma \oplus \mathrm{W}^\pm_{\gamma'}$; cf. proposition 3.8).

The situation becomes even more complicated if we seek good models for the irreducible $\mathrm{o}(g)$ representations $\mathcal{D}^{p,q}$ of bi-spin type (p,q): For instance, if we in the reductions $\mathrm{W}^+_{\gamma_j}$, for $j = 1, \dots, p$, and $\mathrm{W}^-_{\gamma'_k}$, for $k = 1, \cdots, q$, set the light-cone lines equal, so that $\gamma = \gamma_j$, and $\gamma' = \gamma'_k$, then one can take as model for the irreducible representation $\mathcal{D}^{p,q}$, with $p, q \in \frac{1}{2}\mathbb{N}$, the tensor

$$\mathrm{S}^{2p}(\mathrm{W}^+_\gamma) \otimes \mathrm{S}^{2q}(\mathrm{W}^-_{\gamma'}) \tag{6.47}$$

where the symbol S^l stands for the symmetric tensors of degree l, but in general, if the γ_j and γ'_k are allowed to be distinct, there is no such obvious irreducible representation model[49]. Instead, one will have to rely[50] on a Clebsch-Gordan decomposition, which heavily depends on choices of vectorspace basises.

In this context, note that if one attempts to use as spin models the symmetric

[48] Strictly speaking, γ is a cotensor which at each point $a \in M$ is on the the light-cone stalk $\mathcal{L}_a$, defined only up to equivalence with multiplication of an element in $\mathbb{R}^\times$; one could therefore write $\gamma \in \mathcal{L}/\mathbb{R}^\times$.

[49] The treatment in common quantum physics disregards the kind of additional data discussed here. One important discussion is whether such additional data have physical significance; clearly it has practical significance for the computation of a suitable generalization of the so called Einstein equation in general relativity, to include particles with spin.

[50] There is also the possibility of using suitable projectivizations of the representations $V^\pm$, and their corresponding higher tensorproducts; basically, on $V^\pm$, you divide out with the actions $\exp(\mathrm{i}\,\theta\sigma^u_\mp)$, cf. formulas (2.3), (2.27) and (2.29). However, you then end up with the problem of relating this to other data, in quantum mechanics and in general relativity, so I decided to avoid this approach for the time being.

spaces

$$\mathrm{S}^{2p}(V^+)\otimes \mathrm{S}^{2q}(V^-) \tag{6.48}$$

with $p,q\in\frac{1}{2}\mathbb{N}$, then, as $V^\pm=\mathrm{W}^\pm_{\gamma_\pm}\oplus\mathrm{W}^\pm_{\gamma'_\pm}$ (cf. proposition 3.8) for any two pairs of distinct lines on the light-cone γ_+,γ'_+ and γ_-,γ'_- respectively, then one ends up with

$$\begin{aligned}\mathrm{S}^{2p}(V^+)\otimes \mathrm{S}^{2q}(V^-) = \bigoplus_{k+l=p}\mathrm{S}^{2k}(\mathrm{W}^+_{\gamma_+})\otimes \mathrm{S}^{2l}(\mathrm{W}^+_{\gamma'_+})\otimes\\ \bigoplus_{m+n=q}\mathrm{S}^{2m}(\mathrm{W}^-_{\gamma_-})\otimes \mathrm{S}^{2n}(\mathrm{W}^-_{\gamma'_-})\end{aligned} \tag{6.49}$$

with $k,l,m,n\in\frac{1}{2}\mathbb{N}$ then, and where each one of the tensors

$$\begin{cases}\mathrm{S}^{2k}(\mathrm{W}^+_{\gamma_+})\otimes \mathrm{S}^{2l}(\mathrm{W}^+_{\gamma'_+}), & \text{with } k+l=p\\ \mathrm{S}^{2m}(\mathrm{W}^-_{\gamma_-})\otimes \mathrm{S}^{2n}(\mathrm{W}^-_{\gamma'_-}), & \text{with } m+n=q\end{cases} \tag{6.50}$$

are tensorproducts of irreducible $\mathrm{o}(g)_\pm$ representations, and therefore, using Clebsch-Gordan decomposition, fall into sums of irreducible representations, just as in the case of formula (5.27).

So for higher spin, there results a rather complicated situation.

The $\mathrm{o}(g)$ representations $V^\pm$ were defined (equation (6.43) above) by arbitrarily truncating the action of $\mathrm{o}(g)_\mp$ to zero. We can now show that these representations can be naturally extricated using Clifford algebra techniques, by identifying $\mathrm{o}(g)=\Omega^2=\mathrm{C}(g)^2$ via formula (5.23), identifying $V^\pm={}_\pm\mathrm{C}(g)^{\mathrm{odd}}$ as described below, and having $\mathrm{C}(g)^{\mathrm{even}}$ acting on ${}_\pm\mathrm{C}(g)^{\mathrm{odd}}$ by the usual Clifford multiplication.

The isomorphism between $V^\pm$ and ${}_\pm\mathrm{C}(g)^{\mathrm{odd}}$ is defined[51] by

$$\begin{cases}V^\pm \overset{\cong}{\longleftrightarrow} {}_\pm\mathrm{C}(g)^{\mathrm{odd}}\\ \quad x\to x\pm \mathrm{i}\,\omega\,{\scriptstyle\square}\, x = 2p_\pm{\scriptstyle\square}\, x\\ \alpha_1\leftarrow\alpha=\alpha_1+\alpha_3\end{cases} \tag{6.51}$$

where α_1,α_3 refers to the degree one and three components of α considered as an element of $\Omega^{\mathrm{odd}}=\Omega^1\oplus\Omega^3$. The arrows of different directions in formula (6.51) are inverses of each other: First, if $\alpha\in{}_\pm\mathrm{C}(g)^{\mathrm{odd}}$, then $\alpha=p_\pm{\scriptstyle\square}\,\alpha$, and using the decomposition into degrees $\alpha=\alpha_1+\alpha_3$, this equation becomes

$$\alpha_1=\pm\,\mathrm{i}\,\omega\,{\scriptstyle\square}\,\alpha_3 \qquad\Longleftrightarrow\qquad \alpha_3=\pm\,\mathrm{i}\,\omega\,{\scriptstyle\square}\,\alpha_1 \tag{6.52}$$

[51] One would think that it would be more natural sending $x\to p_\pm{\scriptstyle\square}\, x$ in formula (6.51). However, it is more natural to write, for $(x,y)\in V_{\mathbb{C}}\oplus V_{\mathbb{C}}$, $(x,y)=\frac{1}{2}(x+y,x+y)+\frac{1}{2}(x-y,-(x-y))\in V^+\oplus V^-$, giving rise to isomorphisms $(x,y)\to(\frac{1}{2}(x+y),\frac{1}{2}(x-y))\in V^+\oplus V^-$ and $(a,b)\in V^+\oplus V^-\to(a+b,a-b)$. Combined with the isomorphism in formula (6.54) below, this gives the indicated $x\to(1\pm\mathrm{i}\,\omega){\scriptstyle\square}\, x$ of formula (6.51).

Therefore any such $\alpha \in {}_{\pm}\mathrm{C}(g)^{\mathrm{odd}}$ can be written

$$(6.53) \qquad \begin{cases} \alpha = (1 \pm \mathrm{i}\,\omega) \mathbin{\scriptstyle\square} \alpha_1 = 2p_{\pm} \mathbin{\scriptstyle\square} \alpha_1 \\ \quad = (1 \pm \mathrm{i}\,\omega) \mathbin{\scriptstyle\square} \alpha_3 = 2p_{\pm} \mathbin{\scriptstyle\square} \alpha_3 \end{cases}$$

It is clear enough now that $x \mapsto x \pm \mathrm{i}\,\omega \mathbin{\scriptstyle\square} x \mapsto (x \pm \mathrm{i}\,\omega \mathbin{\scriptstyle\square} x)_1 = x$, but then also $\alpha \mapsto \alpha_1 \mapsto (1 \pm \mathrm{i}\,\omega) \mathbin{\scriptstyle\square} \alpha_1 = \alpha$, showing that the two maps are inverses of each other.

We see here that the spin representations $V^{\pm}$ lie on the diagonal in $\mathrm{C}(g)^{\mathrm{odd}}$, and are thus of mixed tensor degree.

Let us compare this to the case of the usual[52] spin representation of the Clifford algebra $\mathrm{C}(g)$, in which case the representation space[53] is the (complex dimension four) exterior algebra $\bigwedge^{\cdot} W$ of some maximal isotropic vectorspace W. Here, spin $(\frac{1}{2}, 0)$ say, consists of tensors of degree 1 (complex dimension two), and then spin $(0, \frac{1}{2})$ consists of tensors of degrees 0 and 2 (complex dimension two).

If this spin representation is restricted down to $\mathrm{C}(g)^{\mathrm{even}}$, then $\mathrm{C}(g)^{\mathrm{even}}$ leaves the set of odd degree elements invariant, so it acts on the isotropic vectorspace W chosen. Since it is known that the complex Clifford algebra $\mathrm{C}(g)^{\mathrm{even}}_{\mathbb{C}}$ has only two irreducible representations, from the point of view of abstract algebra, we must by necessity end up with one of the spin representations $(\frac{1}{2}, 0)$ or $(O, \frac{1}{2})$. However, when worked out in tensors, the tensor formula for the $\mathrm{C}(g)^{\mathrm{even}}$ representation onto W is not the same as the one described in sections 2 and 3; in fact, you get formulas which are difficult to treat at all, formulas which in addition depends on the choice of a complementary isotropic vectorspace W', (complementary meaning that $V_{\mathbb{C}} = W \oplus W'$). For such reasons, in my experience, these classical representation formulas seem to be a dead end.

The explanation of this phenomenon, depends on the fact that there is additional information in the choice of the representations, for instance, such things as how the representations link up to the metric. This kind of information may thus alternatively be considered as being mathematical or physical, then.

Note also that the $\mathrm{C}(g)^{\mathrm{even}}$ representations ${}_{\pm}\mathrm{C}(g)^{\mathrm{odd}}$ do not extend to representations of the $\mathrm{C}(g)$ module. From the point of view of physics this is an interesting situation, because even though the electromagnetic two-form F is in $\mathrm{C}(g)^{\mathrm{even}}$, so is the electromagnetic potential A not in $\mathrm{C}(g)^{\mathrm{odd}}$; one will have to rely on the Infeldt-van der Waerden symbols mentioned in the beginning of section 4 then (see footnote 35). On the other hand, the use of the electromagnetic potential A in the Dirac equation forces the simultaneous presence of both spin and antispin types, which does not conform with observation; electrons, for instance, can exist without positrons being nearby.

[52]This leads to the representation of the Dirac equation called the chiral (χ-ral) representation by physicists, with the upper/lower components of the chiral representation corresponding to tensors of odd/even degree, cf. equation (1.11) in the introduction.

[53]See Bourbaki [4], ch IX, for this algebraic construction.

There remains to show that the action of $S((1 \pm \mathrm{i}\,\omega) \,\square\, F)$ on $V^{\pm}$ is the same as that of the corresponding element in $\mathrm{C}(g)^{\mathrm{even}}$ acting on ${}_{\pm}\mathrm{C}(g)^{\mathrm{odd}}$, when identified using the isomorphism described in equation (6.51); also recall that in section 4, the identification $\mathrm{C}(g)^2 = \Omega^2$ was made. The method to show this is by determining these actions through explicit computations.

So set up an isomorhism between $\mathrm{C}(g)^{\mathrm{odd}} = \Omega^1 \oplus \Omega^3$ and $V_{\mathbb{C}} \oplus V_{\mathbb{C}}$ by

$$\begin{cases} \mathrm{C}(g)^{\mathrm{odd}} \overset{\cong}{\longleftrightarrow} V_{\mathbb{C}} \oplus V_{\mathbb{C}} \\ \alpha = \alpha_1 + \alpha_3 \rightarrow (\alpha_1, \mathrm{i}\,\omega \,\square\, \alpha_3) \\ x + \mathrm{i}\,\omega \,\square\, y \leftarrow (x, y) \end{cases} \tag{6.54}$$

and, using the fact that $(\mathrm{i}\,\omega)^2 = 1$, it is easy enough to verify that the arrows of different directions are inverses of each other.

This enables us to describe $\mathrm{C}(g)^{\mathrm{odd}}$ as the set of pairs (x, y), with $x, y \in V_{\mathbb{C}}$, and the next step is to give explicit formulas[54] for the Clifford multiplication $\square$ of $\mathrm{C}(g)^{\mathrm{even}}$ on the module $V_{\mathbb{C}} \oplus V_{\mathbb{C}}$ consisting of such pairs (x, y). If $a, b \in \mathrm{C}(g)^1 = \Omega^1$, and $\lambda \in \mathrm{C}(g)^0 = \Omega^0$, one finds that

$$\lambda(x, y) = (\lambda x, \lambda y) \tag{6.55}$$

$$\begin{cases} a \,\square\, b \,\square\, (x, 0) = ((g(a, b) + S(a \wedge b)).x, S(\mathrm{i}\,\omega \,\square\, (a \wedge b)).x) \\ a \,\square\, b \,\square\, (0, y) = (S(\mathrm{i}\,\omega \,\square\, (a \wedge b)).y, (g(a, b) + S(a \wedge b)).y) \end{cases} \tag{6.56}$$

$$\mathrm{i}\,\omega \,\square\, (x, y) = (y, x) \tag{6.57}$$

In equation (6.56), it important to note that $a \wedge b = a \,\square\, b - g(a, b)$, by the conventions set up in section 4.

In order to show these formulas, equation (6.55) is a mere trifle and is clearly there only for reasons of completeness. It is also quite plain to show equation (6.57), since[55] one has $\mathrm{i}\,\omega \,\square\, (x, y) \leftrightarrow \mathrm{i}\,\omega \,\square\, (x + \mathrm{i}\,\omega \,\square\, y) = (y + \mathrm{i}\,\omega \,\square\, x) \leftrightarrow (y, x)$ under the isomorphism (6.54).

Then the second equation in (6.56) follows from the first, and from formula (6.57), since then

$$\begin{aligned} a \,\square\, b \,\square\, (0, y) &= \mathrm{i}\,\omega \,\square\, a \,\square\, b \,\square\, \mathrm{i}\,\omega \,\square\, (0, y) = \mathrm{i}\,\omega \,\square\, a \,\square\, b \,\square\, (y, 0) \\ &= \mathrm{i}\,\omega \,\square\, ((g(a, b) + S(a \wedge b)).y, S(\mathrm{i}\,\omega \,\square\, (a \wedge b)).y) \\ &= (S(\mathrm{i}\,\omega \,\square\, (a \wedge b)).y, (g(a, b) + S(a \wedge b)).y) \end{aligned}$$

Then there remains to show the first equation of formula (6.56), which requires a more detailed investigation of the operator $S(a \wedge b)$.

[54] Strictly speaking, the $\mathrm{C}(g)^{\mathrm{even}}$ module structure on $V_{\mathbb{C}} \oplus V_{\mathbb{C}}$ is defined by the $\mathrm{C}(g)^{\mathrm{even}}$ action on $\mathrm{C}(g)^{\mathrm{odd}}$ transported by the isomorphism in equation (6.54); the equations (6.55), (6.56) and (6.57) then give explicit formulas for this $\mathrm{C}(g)^{\mathrm{even}}$ module structure.

[55] In this context, the arrow $\leftrightarrow$ is pronounced *corresponds to*, referring to correspondence under the isomorphism in formula (6.54) then.

So, in order to show the first equation of formula (6.56), first note that equation (4.23) can be rewritten

$$a \mathbin{\scriptscriptstyle\square} b \mathbin{\scriptscriptstyle\square} x = a \wedge b \wedge x + (S(a \wedge b) + g(a, b)).x \tag{6.58}$$

since $a \mathbin{\scriptscriptstyle\square} b \mathbin{\scriptscriptstyle\square} x = a \wedge b \wedge x + g(b, x)a - g(a, x)b + g(a, b)x = a \wedge b \wedge x + S(a \wedge b).x + g(a, b)x$, also invoking formula (5.23).

Now we want to show that

$$a \wedge b \wedge x = \mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} S(\mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} (a \wedge b)).x \tag{6.59}$$

This will follow if we show that

$$\mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} (a \wedge b \wedge x) = S(\mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} (a \wedge b)).x \tag{6.60}$$

clearly, since $(\mathrm{i}\,\omega)^2 = 1$.

In order to show equation (6.60), first note that for any two-form $F \in \Omega^2 = \mathrm{C}(g)^2$, one has

$$S(F).x = \tfrac{1}{2}[F, x]_{\scriptscriptstyle\square} \tag{6.61}$$

where $[F, x]_{\scriptscriptstyle\square} := F \mathbin{\scriptscriptstyle\square} x - x \mathbin{\scriptscriptstyle\square} F$ denotes the Clifford multiplication commutator. This follows, because if $F = a \mathbin{\scriptscriptstyle\square} b$ (which is also equal to $a \wedge b$, since F was assumed to be in Ω^2), then, using the notation $\{x, y\}_{\scriptscriptstyle\square} := x \mathbin{\scriptscriptstyle\square} y + x \mathbin{\scriptscriptstyle\square} y$ for the Clifford multiplication commutator (which by definition is equal to $g(x, y)$), one has $\frac{1}{2}[F, x]_{\scriptscriptstyle\square} = \frac{1}{2}[a \mathbin{\scriptscriptstyle\square} b, x]_{\scriptscriptstyle\square} = \frac{1}{2}((a \mathbin{\scriptscriptstyle\square} b \mathbin{\scriptscriptstyle\square} x + a \mathbin{\scriptscriptstyle\square} x \mathbin{\scriptscriptstyle\square} b) - (a \mathbin{\scriptscriptstyle\square} x \mathbin{\scriptscriptstyle\square} b + x \mathbin{\scriptscriptstyle\square} a \mathbin{\scriptscriptstyle\square} b)) = a \mathbin{\scriptscriptstyle\square} \frac{1}{2}\{b, x\}_{\scriptscriptstyle\square} - \frac{1}{2}\{a, x\}_{\scriptscriptstyle\square} \mathbin{\scriptscriptstyle\square} b = ag(b, x) - g(a, x)b = S(a \wedge b).x$, and the case of a general two-form follows by linear extension.

Then, starting on the right hand side of equation (6.60), and writing $F := a \wedge b$, we get[56]

$$\begin{aligned}
S(\mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} F).x = \tfrac{1}{2}[\mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} F, x]_{\scriptscriptstyle\square} &= \tfrac{1}{2}(\mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} F \mathbin{\scriptscriptstyle\square} x - x \mathbin{\scriptscriptstyle\square} \mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} F) = \mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} \tfrac{1}{2}(F \mathbin{\scriptscriptstyle\square} x + x \mathbin{\scriptscriptstyle\square} F) \\
&= \mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} \tfrac{1}{2}(F.(\overleftarrow{e}(x) - \overleftarrow{\imath}(x)) + (e(x) - i(x)).F) \\
&= \mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} \tfrac{1}{2}(F \wedge x + x \wedge F - (F.\overleftarrow{\imath}(x) + i(x).F)) \\
&= \mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} (F \wedge x)
\end{aligned}$$

where the last line follows by equation (5.16), since $F.\overleftarrow{\imath}(x) = (-1)^{\deg F - 1} i(x).F = -i(x).F$. So this shows equation (6.60), and therefore also equation (6.59).

Returning now to the proof of the first equation in formula (6.56), first note that equation (6.58), using equation (6.59), can be rewritten

$$a \mathbin{\scriptscriptstyle\square} b \mathbin{\scriptscriptstyle\square} x = (g(a, b) + S(a \wedge b)).x + \mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} S(\mathrm{i}\,\omega \mathbin{\scriptscriptstyle\square} (a \wedge b)).x \tag{6.62}$$

[56] Concerning Clifford multiplication from the right, see comment in paragraph after equation (5.18).

From this, the first equation in formula (6.56) immediately follows, since then

$$\begin{aligned} a\circ b\circ(x,0) &\leftrightarrow a\circ b\circ x = (g(a,b)+S(a\wedge b)).x + \mathrm{i}\,\omega\circ S(\mathrm{i}\,\omega\circ(a\wedge b)).x \\ &\leftrightarrow ((g(a,b)+S(a\wedge b)).x, (\mathrm{i}\,\omega)^2 S(\mathrm{i}\,\omega\circ(a\wedge b)).x) \\ &= ((g(a,b)+S(a\wedge b)).x, S(\mathrm{i}\,\omega\circ(a\wedge b)).x) \end{aligned}$$

where $\leftrightarrow$ denotes correspondence under the isomorphism indicated in formula (6.54).

One can also prove the formula (6.56) by computations that are more explicit, by letting the elements a, b run over the elements in a restricted orthonormal basis ω^μ, and dividing into cases depending whether $x \in (a,b)_{\mathbb{C}}$ or $x \perp a, b$.

It is now easy to compute the action of the Clifford bundle $\mathrm{C}(g)^{\text{even}}$ on the modules $V^\pm \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}} \cong \mathrm{C}(g)^{\text{odd}}$.

First we see that

$$V^\pm = \{(x, \pm x) \in V_{\mathbb{C}} \oplus V_{\mathbb{C}} \mid x \in V_{\mathbb{C}}\} \tag{6.63}$$

clearly, since the elements in ${}_\pm\mathrm{C}(g)^{\text{odd}}$ by equation (6.53) can be written on the form $(1 \pm \mathrm{i}\,\omega)\circ x = x \pm \mathrm{i}\,\omega\circ x$, for some element $x \in \mathrm{C}(g)^1_{\mathbb{C}} = V_{\mathbb{C}}$, and, under the isomorphism indicated in formula (6.54), this element maps to the element $(x, \pm(\mathrm{i}\,\omega)^2\circ x) = (x, \pm x) \in V_{\mathbb{C}} \oplus V_{\mathbb{C}}$.

Next, if now $(x, \pm x) \in V^\pm$, for some $x \in V_{\mathbb{C}}$, one finds, for two choices of signs $\varepsilon, \eta = +, -$, and using equation (6.57), that

$$\begin{aligned} p_\varepsilon\circ(x,\eta x) &= \tfrac{1}{2}(1+\mathrm{i}\,\varepsilon\omega)\circ(x,\eta x) = \tfrac{1}{2}((x,\eta x)+\varepsilon(\eta x, x)) \\ &= \tfrac{1}{2}(1+\varepsilon\eta)(x,\eta x) \end{aligned}$$

or, summarizing,

$$p_\varepsilon\circ(x,\eta x) = \tfrac{1}{2}(1+\varepsilon\eta)(x,\eta x) = \begin{cases} (x,\eta x), & \varepsilon\eta = 1 \\ 0, & \varepsilon\eta = -1 \end{cases} \tag{6.64}$$

This can also be written more explicitly as

$$\begin{cases} p_\pm\circ(x,\pm x) = (x,\pm x) \\ p_\mp\circ(x,\pm x) = 0 \end{cases} \tag{6.65}$$

confirming what we already know, namely that the embedded $V^\pm \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}}$ vectorbundles are $\mathrm{C}(g)^{\text{even}}_\pm$ modules.

If now $a, b \in \mathrm{C}(g)^1$, then, using formula (6.56), one gets

$$\begin{aligned} a\circ b\circ(x,\pm x) &= a\circ b\circ(x,0) \pm a\circ b\circ(0,x) \\ &= (g(a,b)x + S(a\wedge b).x, S(\mathrm{i}\,\omega\circ(a\wedge b)).x) \\ &\qquad \pm (S(\mathrm{i}\,\omega\circ(a\wedge b)).x, g(a,b)x + S(a\wedge b).x) \\ &= g(a,b)(x,\pm x) + S((1\pm\mathrm{i}\,\omega)\circ(a\wedge b)).(x,\pm x) \end{aligned} \tag{6.66}$$

where, in the last line, the shorthand notation $S(F).(x, y) := (S(F).x, S(F).y)$, for $x, y \in V_{\mathbb{C}}$ and $F \in \Omega^2$, is used.

If we now use the identity $a \circ b = a \wedge b + g(a, b)$, then equation (6.66) transforms into

$$(a \wedge b) \circ (x, \pm x) = S((1 \pm \mathrm{i}\,\omega) \circ (a \wedge b)).(x, \pm x) \tag{6.67}$$

or, somewhat more generally, for any $F \in \mathrm{C}(g)^2$,

$$F \circ (x, \pm x) = S((1 \pm \mathrm{i}\,\omega) \circ F).(x, \pm x) \tag{6.68}$$

which follows by linear extension in the variable F.

This, in fact, leads to what we wanted to show: If $\leftrightarrow$ denotes the isomorphism $V^{\pm} \leftrightarrow V_{\mathbb{C}} \oplus V_{\mathbb{C}}$, with $x \leftrightarrow (x, \pm x)$, which is also the combination of the isomorphisms in formulas (6.51) and (6.54), then from equation (6.43) we see that

$$\begin{aligned} \rho_{\pm}(F).x \longleftrightarrow & \rho_{\pm}(F).(x, \pm) \\ &= S((1 \pm \mathrm{i}\,\omega) \circ F).(x, \pm x) \\ &= F \circ (x, \pm) \end{aligned} \tag{6.69}$$

In other words, the Lie bundle $\mathrm{o}(g)$ representation structure on the vectorbundles $V^{\pm}$ as described by equation (6.43) agrees with the corresponding Clifford algebra multiplication when $V^{\pm}$ is embedded on the diagonal in $\mathrm{C}(g)^{\mathrm{odd}}$ according to the formula (6.51).

Combining with equations (6.41) and (6.44), we find

$$\begin{aligned} \sigma_{\pm}^{F}.x &\longleftrightarrow \varepsilon_{\sigma} F \circ (x, \pm x) \in V_{\mathbb{C}} \oplus V_{\mathbb{C}} \\ &\longleftrightarrow \varepsilon_{\sigma} F \circ (1 \pm \mathrm{i}\,\omega) \circ x \in \mathrm{C}(g)^{\mathrm{odd}} \end{aligned} \tag{6.70}$$

under the assumption that $x \leftrightarrow (x, \pm x) \leftrightarrow x \pm \mathrm{i}\,\omega \circ x$, using the isomorphisms in formulas (6.51) and (6.54), of course.

A Lorentz Vector Calculus

The formulas below are useful in making the transition back and forth between the time-space picture, and the genuine Lorentz picture. There are similar types of formulas in use among physisists, but the main differences here are that the formulas below are closer to coordinate-free notation, and that they are specific to what is tensors and cotensors in a differential geometric setting.

Let x^μ be a set of local coordinates at some point a on the Lorentz-manifold M, such that $\omega^\mu := (\omega^\mu)_{\mu=0,1,2,3} := (dx^\mu)_{\mu=0,1,2,3}$ is a restricted orthonormal basis. Then one also has $cdt = dx^0$, and $\frac{1}{c}\frac{d}{dt}$, where, as usual, the constant c is the speed of light.

It will be convenient to write $\tau := \omega^0$, and $\boldsymbol{\omega} := (\omega^j)_{j=1,2,3}$.

The volume element will be denoted by $\omega := \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3$.

For an even permutation (j,k,l) of $(1,2,3)$, write

$$\text{(A.1)} \qquad \begin{cases} \tilde{\omega}^j := \omega^k \wedge \omega^l \\ \tilde{\boldsymbol{\omega}} := (\tilde{\omega}^j)_{j=1,2,3} \end{cases}$$

If $f \in \Omega^0$, that is, f is a function on the manifold in question, write $\boldsymbol{\nabla} f := \frac{\partial f}{\partial x^j} dx^j$, so that then

$$\text{(A.2)} \qquad df = \frac{df}{dt}dt + \boldsymbol{\nabla} f = \frac{\partial f}{\partial x^0}\tau + \boldsymbol{\nabla} f$$

In the subsequent formulas, put $c = 1$; if one wants to have the constant c back, one needs only to do the replacement $t \to ct$.

For a one-form $a \in \Omega^1$, written on the form $a := \alpha\tau + \mathbf{a}$, one has the following two equations:

$$\text{(A.3)} \qquad da = \tau \wedge (\frac{d}{dt}\mathbf{a} - \boldsymbol{\nabla}\alpha) + (\boldsymbol{\nabla} \times \mathbf{a}) \cdot \tilde{\boldsymbol{\omega}} \in \Omega^2$$

$$\text{(A.4)} \qquad \operatorname{div} a = \varepsilon_\tau(\frac{d\alpha}{dt} - \boldsymbol{\nabla} \cdot \mathbf{a}) \in \Omega^0$$

Here ε_τ is the sign of the temporal part of the metric.

One will have to do some interpretation of formulas of this kind: For instance, in the equation (A.3), working out in coordinates, it is obvious what $\boldsymbol{\nabla}\times\mathbf{a}$ should mean; one can also write $(\boldsymbol{\nabla} \times \mathbf{a}) \cdot \tilde{\boldsymbol{\omega}} = \boldsymbol{\nabla} \wedge \mathbf{a}$.

For a two-form $f \in \Omega^2$, written on the form

$$\text{(A.5)} \qquad f := \tau \wedge \mathbf{a} + \mathbf{b} \cdot \tilde{\boldsymbol{\omega}} = a_j\omega^0 \wedge \omega^j + b_j\tilde{\omega}^j = \tau \wedge \mathbf{a} + \omega \,\square\, \tau \,\square\, \mathbf{b}$$

one has the following two equations:

$$\text{(A.6)} \qquad df = \tau \wedge ((\frac{d}{dt}\mathbf{b} - \boldsymbol{\nabla} \times \mathbf{a}) \cdot \tilde{\boldsymbol{\omega}}) + (\boldsymbol{\nabla} \cdot \mathbf{b})\omega_{\hat{\tau}} \in \Omega^3$$

$$\text{(A.7)} \qquad \operatorname{div} f = \varepsilon_\tau((\boldsymbol{\nabla} \cdot \mathbf{a})\tau + (\boldsymbol{\nabla} \times \mathbf{b} + \frac{d}{dt}\mathbf{a})) \in \Omega^1$$

For elements $a, b, c \in \Omega^1$, write $a := a_0\tau + \mathbf{a} := a_0\omega^0 + a_j\omega^j$, etc. Then one has the following formulas:

$$a \wedge b = (a_0\tau + \mathbf{a}) \wedge (b_0\tau + \mathbf{b}) = \tau \wedge (a_0\mathbf{b} - b_0\mathbf{a}) + (\mathbf{a} \times \mathbf{b}) \cdot \tilde{\boldsymbol{\omega}} \tag{A.8}$$

$$a \wedge b \wedge c = \tau \wedge ((a_0\mathbf{b} \times \mathbf{c} + b_0\mathbf{c} \times \mathbf{a} + c_0\mathbf{a} \times \mathbf{b}) \cdot \tilde{\boldsymbol{\omega}}) + \det(\mathbf{a}, \mathbf{b}, \mathbf{c})\omega_{\hat{\tau}} \tag{A.9}$$

and where also $\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

If one has four vectors $a, b, c, d \in \Omega^1$, the corresponding formula for $a \wedge b \wedge c \wedge d$ will just be the expansion of the determinant $\det(a, b, c, d)$ along the $\tau = \omega^0$ row.

For elements $a \in \Omega^1$, $a = a_0\tau + \mathbf{a}$, and $f \in \Omega^2$, $f = \tau \wedge \mathbf{b} + \mathbf{c} \cdot \tilde{\boldsymbol{\omega}}$, one has the following formula:

$$a \wedge f = (a_0\tau + \mathbf{a}) \wedge (\tau \wedge \mathbf{b} + \mathbf{c} \cdot \tilde{\boldsymbol{\omega}}) = \tau \wedge (a_0\mathbf{c} - \mathbf{a} \times \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})\omega_{\hat{\tau}} \tag{A.10}$$

For the exterior product of two elements in Ω^2, there is the following formula:

$$(\tau \wedge \mathbf{a} + \mathbf{b} \cdot \tilde{\boldsymbol{\omega}}) \wedge (\tau \wedge \mathbf{c} + \mathbf{d} \cdot \tilde{\boldsymbol{\omega}}) = (\mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c})\omega \tag{A.11}$$

Note in particular, writing $f = \tau \wedge \mathbf{a} + \mathbf{b} \cdot \tilde{\boldsymbol{\omega}}$, that

$$f \wedge f = 2\mathbf{a} \cdot \mathbf{b}\,\omega \tag{A.12}$$

The Hodge star dual of f is then

$$*f = *(\tau \wedge \mathbf{a} + \mathbf{b} \cdot \tilde{\boldsymbol{\omega}}) = \tau \wedge \mathbf{b} + (-\mathbf{a}) \cdot \tilde{\boldsymbol{\omega}} \tag{A.13}$$

If one is using Clifford multiplication, then $\omega \mathbin{\scriptstyle\square} f = - * f$, so that formula (A.13) becomes

$$\omega \mathbin{\scriptstyle\square} f = \tau \wedge (-\mathbf{b}) + \mathbf{a} \cdot \tilde{\boldsymbol{\omega}} \tag{A.14}$$

The corresponding formulas involving the metric then becomes as follows:

$$q(f) = -\mathbf{a}^2 + \mathbf{b}^2 \tag{A.15}$$

$$g(f, *f) = -2\mathbf{a} \cdot \mathbf{b} \tag{A.16}$$

For a two-form $f \in \Omega^2$ on the form $f := \tau \wedge \mathbf{a} + \mathbf{b} \cdot \tilde{\boldsymbol{\omega}}$, and a vectorfield $U := U^0 \frac{\partial}{\partial x^0} + U^j \frac{\partial}{\partial x^j} = U^0 \frac{\partial}{\partial x^0} + \mathbf{U}$ one has the following formula

$$\begin{aligned} f(U) &= (\mathbf{U} \cdot \mathbf{a})\tau + (-U^0\mathbf{a} + \mathbf{U} \times \mathbf{b}) \cdot \boldsymbol{\omega} \\ &= (U^j a_j)\omega^0 - U^0 a_j \omega^j + \sum \varepsilon_{jkl} U^j b_k \omega^l, \end{aligned} \tag{A.17}$$

and note that despite the unusual constellation of upper and lower indices, this formula does not involve any metric conversions.

B Notation

Greek indices μ, ν, ... , run over the integers 0, 1, 2, 3, where 0 is the index of the temporal part. Latin indices j, k, l, ... , run over the integers 1, 2, 3, being the indices of the spatial part.

For the most of the time there is a summation convention in effect, that an index which appears in two positions (usually as a superscript and a subscript) normally should be summed, for example $a_\mu b^\mu := \sum_{\mu=0,1,2,3} a_\mu b^\mu$, and $a_j b_j := \sum_{j=1,2,3} a_j b_j$.

In formulas, and often in sentences and paragraphs, if there appear double signs $\pm$ and $\mp$, this stands for the two cases defined by making one of the choices upper/lower throughout the reasoning; the context will tell.

The Lorentz metric is denoted by $g = g_{\mu\nu} dx^\mu \otimes dx^\nu =: g_{\mu\nu}$, and is of type $+---$ or of type $-+++$. The metric g is a symmetric tensor field of type $(0,2)$. The inverse at each point of the matrix $(g_{\mu\nu})_{\mu\nu}$ defines a (2,0) tensor field denoted by $g^{-1} = (g^{-1})^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} =: (g^{-1})^{\mu\nu}$; in situations which do not involve metric variations, the notation $g^{\mu\nu} := (g^{-1})^{\mu\nu}$, dropping the superscript -1, is also used, since one then can use tensor shifting conventions as usual.

By ε_τ is meant the sign of the temporal part of the metric g; if g is of type $\pm\mp\mp\mp$, then $\varepsilon_\tau = \pm$. And ε_σ denotes the sign of the spatial part of the metric g; if g is of type $\pm\mp\mp\mp$, then $\varepsilon_\sigma = \mp$.
Note: One always has $\varepsilon_\tau \varepsilon_\sigma = -1$.

The notation q is for the quadratic form associated with the metric g; at each point of the Lorentz-manifold studied, one has $q(x) := g(x,x)$, where x is the stalk-value of a vector-field for a start, but this formula is then being extended to other tensors as the metric g is extended to such tensors. The main use of q is notational simplification.

The Pauli spin matrices are

$$\sigma^0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

which are the ones most commonly used in quantum physics today. The other type is then the complex conjugates

$$(\sigma^\mu)^{\mathrm{cx}} = \begin{cases} \sigma^\mu, & \text{if } \mu \neq 2 \\ -\sigma^2, & \text{if } \mu = 2 \end{cases}$$

Strictly speaking, it is only the matrices σ^j and $(\sigma^j)^{\mathrm{cx}}$, for $j = 1, 2, 3$, that have anything to do with spin, but it proves convenient to add $\sigma^0 := 1$ as well, for the use in certain formulas.

If A is a ring, $A^\times$ is the notation for the set of invertible elements in A, that is, the set of elements $a \in A$ for which there are elements $u, v \in A$ satisfying $au = va = 1$; it then follows that $u = v$, and one writes $a^{-1} := u = v$. For example, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ is the set of non-zero real and complex numbers, respectively, and $\mathbb{Z}^\times = \{+1, -1\}$.

A triple (j, k, l) often stands for an even permutation of $(1, 2, 3)$; when this is being the case, one has the notation $\tilde{\omega}^j := \omega^k \wedge \omega^l$.

By Lie action (derivation) of a linear operator A on a product $x \cdot y$ of some kind is meant $A(x \cdot y) := (Ax) \cdot y + x \cdot (Ay)$; by group (diagonal) action of the same operator A onto the same product $x \cdot y$ is meant $A(x \cdot y) := (Ax) \cdot (Ay)$. One thus has to carefully watch out for the context, since the notation does not tell whether it is Lie or group action in use.

There is in use the symbol ε_I, where $I = (j_1, \ldots, j_k)$ is a finite ordered tuple consisting of elements in J, a totally ordered set; if the elements $j_1, \ldots, j_k$ are distinct, one sets $\varepsilon_I = (-1)^k$, where k are the number of inversions in the tuple $I = (j_1, \ldots, j_k)$, and if two of the elements $j_1, \ldots, j_k$ are being the same, one sets $\varepsilon_I = 0$. If σ is a permutation of J, only moving a finite number of elements, one writes $\operatorname{sgn} \sigma := \varepsilon_\sigma := \varepsilon_{\sigma(J)}$, for the sign of this permutation.

The complex vectorspace generated by a, b is denoted by $(a, b)_{\mathbb{C}}$, and the real vectorspace generated by a, b is denoted by $(a, b)_{\mathbb{R}}$; this is not to be confused with (a, b), an ordered pair.

The exterior algebra on a vectorspace V is denoted by $\bigwedge V$; and ΩV denotes set of antisymmetrical linear functions on V.

If W, W' are two subvectorspaces contained in the vector space V then the direct sum notation $V = W \oplus W'$ is short for that the conditions $V = W + W'$ and $W \cap W' = 0$ both hold.

The manifold M manifold we are working with in this paper is smooth, that is, coordinate functions can be taken in $\mathcal{C}^\infty$.

Note the difference between $T^{p,q}$, which denotes the space of tensors of type (p, q), versus $T^{p,\cdot}$, which is $\bigoplus_{q=0}^{\infty} T^{p,q}$, but sometimes this distinction is being blurred.

For equations, $A := B$ means that "A is defined by B" or "A is assigned the value B", $A =: B$ means that "A defines B" or "A assigns its value to B", and $A :=: B$ means that "A is by definition equal to B", used to indicate either an alternate notation, or for an defining equation of an implicitly defined quantity.

There is the shorthand notation $f(x).a$ for $f(x)(a) = f(x, a)$.

There are alternative notations $\mathrm{i} :=: \sqrt{-1}$ for the imaginary unit, and note that i in this paper happens to be typset upright (as opposed to i), because it is not a variable, and in order to liberate the symbol i for other use, such as a summation index.

Clifford multiplication is denoted by $\circ$.

Occasionally, the notation $x \mapsto y$, meaning that "x maps to y", is used instead of $x \to y$, for emphasis.

Lie algebra represention are sometimes being called modules, making use of common algebra terminology.

If, in the references below, two years are given of the form 1971/1983, the first year indicates the publishing year of the first edition (which may be a different publisher than the one indicated), and the second year is the publishing year of the copy I used.

If, on the hand, two years are given of the form 1971&1974, it indicates the two different publishing years of different volumes in a two-volume set listed together.

The place of publishing has been excluded, given that today complete information can easily be obtained from computer data bases.

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